# An Introduction to the Approximation of Functions

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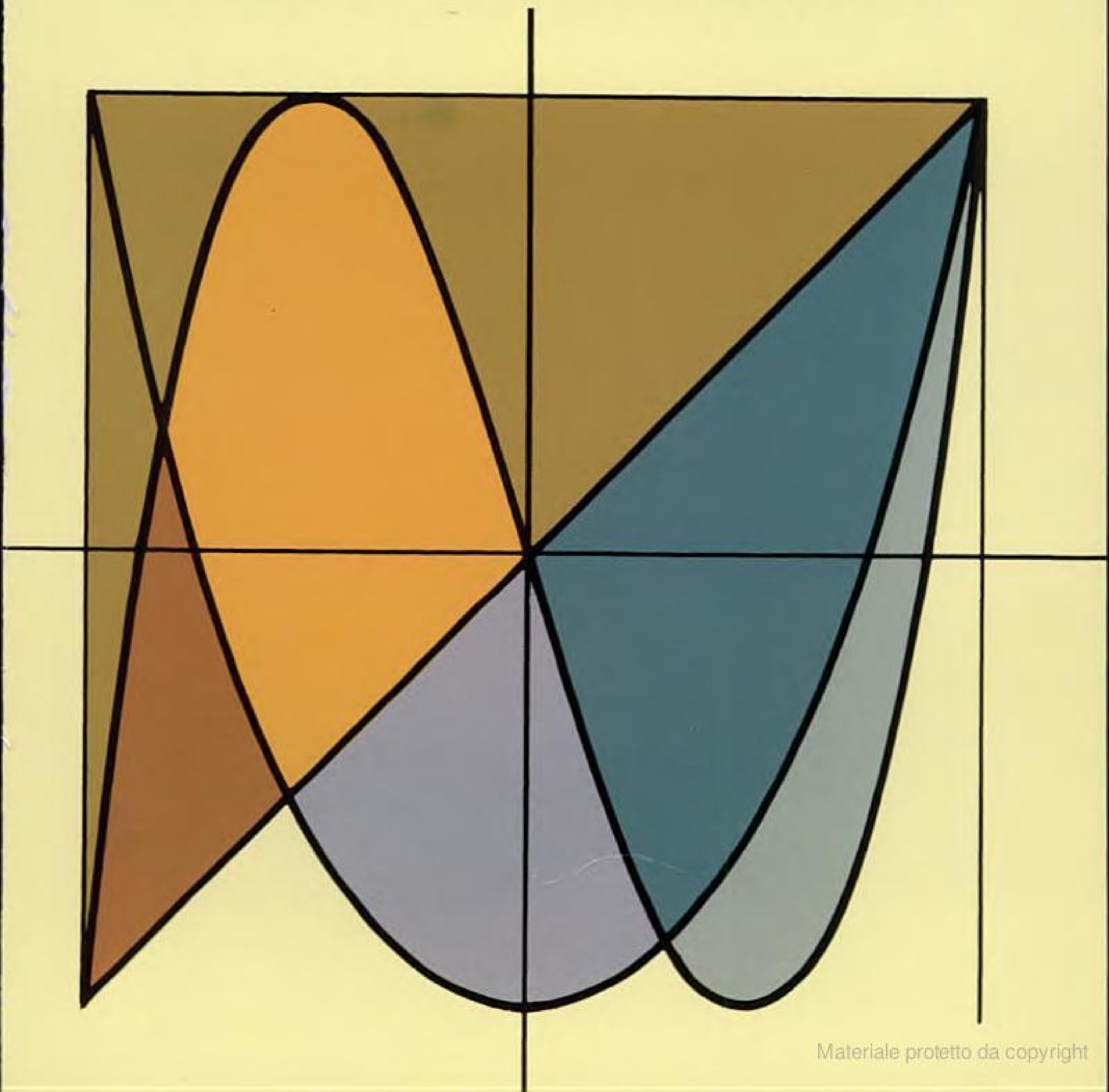

*For*
*Madeline*
*and*
*Jean*



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# INTRODUCTION

There are certain generalities about approximation theory that will be useful in our later, more detailed study of specific approximation techniques. The natural setting for these general results is a *normed linear space*. Linear spaces have become familiar objects in mathematics, and so we assume that the reader is familiar with their definition and most elementary properties. We shall take the scalars to be the real numbers unless some other field is specified.

Let $V$ be a linear space. We recall that a *norm* is a function from $V$ into the nonnegative real numbers. This function is written $\|\cdot\|$ and satisfies the following three properties:

$$\begin{aligned} &\text{(i)} && \|v\| \geq 0 \text{ with equality if and only if } v = 0. \\ &\text{(ii)} && \|\lambda v\| = |\lambda|\,\|v\| \qquad \text{for any scalar } \lambda. \\ &\text{(iii)} && \|v + w\| \leq \|v\| + \|w\| \qquad \text{(the Triangle Inequality).} \end{aligned} \tag{I.1}$$

The norm gives us a notion of *distance* in $V$. If $w, v \in V$, then the distance from $w$ to $v$ (or $v$ to $w$) is $\|v - w\|$.

We are now in a position to present the general setting for much of approximation theory. Let $W$ be a subset of $V$, then, given $v \in V$, the approximation problem, baldly stated, is: Find a $w \in W$ whose distance from $v$ is least; that is, find $w^* \in W$ such that $\|v - w\|$ is least for $w = w^*$. Such a $w^*$ we call a best approximation to $v$ out of $W$. Problems arise immediately. Is there such a $w^*$? If there is, is there only one? Since, as we shall see, many of the most widely studied and used methods of approximation are instances of this general approximation problem, we shall save much duplication of effort by obtaining some results in the general situation.

We turn first to the existence question. We have

THEOREM I.1. *If $V$ is a normed linear space and $W$ a finite-dimensional subspace of $V$, then, given $v \in V$, there exists $w^* \in W$ such that*

$$\|v - w^*\| \leq \|v - w\|$$

*for all $w \in W$.*

*Proof.* Since $0 \in W$, it is a competitor for best approximation to $v$ out of $W$. Its distance from $v$ is $\|v - 0\| = \|v\|$. If $\|v - w\| > \|v\|$, we are, therefore,

polynomials (of any degree). Clearly, $W$ is not finite-dimensional (if it were, what could its dimension be?). We wish to show that $f(x) = 1/(1 - x)$ has no best approximation in the uniform sense on $[0, \frac{1}{2}]$ out of $W$. Note that, given $\varepsilon > 0$, there exists $N$ such that

$$|f(x) - (1 + x + x^2 + \dots + x^N)| < \varepsilon, \qquad 0 \le x \le \tfrac{1}{2}.$$

Hence, if there were a best uniform approximation to $f(x)$ out of $W$, say $p^*$, it would have to satisfy

$$\|f - p^*\| = 0,$$

which implies that

$$\frac{1}{1 - x} \equiv p^*,$$

an impossibility.

Suppose now that $W$ is a subspace of $V$ and let $W^*$ be the set of best approximations to a given $v \in V$ out of $W$. (Theorem I.1 gives us a condition under which $W^*$ is not empty.) We wish to prove that $W^*$ is a *convex* set. We recall that a set, $S$, in a linear space is convex if $s_1, s_2 \in S$ implies that

$$\lambda_1 s_1 + \lambda_2 s_2 \in S$$

if $\lambda_1$ and $\lambda_2$ are nonnegative and

$$\lambda_1 + \lambda_2 = 1.$$

If $S$ is empty or consists of one point, then it is clearly convex.

THEOREM I.2. *If $v \in V$ and $W$ is a subspace of $V$, the set of best approximations to $v$ out of $W$, call it $W^*$, is convex.*

*Proof.* If $W^*$ is empty, the theorem is true. Suppose that $w_1^*, w_2^* \in W^*$; then

$$\|v - w_1^*\| = \|v - w_2^*\| = \rho.$$

Suppose $\lambda_1, \lambda_2 \ge 0$ and $\lambda_1 + \lambda_2 = 1$; then

$$\begin{aligned} \|v - (\lambda_1 w_1^* + \lambda_2 w_2^*)\| &= \|\lambda_1(v - w_1^*) + \lambda_2(v - w_2^*)\| \\ &\le \lambda_1\|v - w_1^*\| + \lambda_2\|v - w_2^*\| = (\lambda_1 + \lambda_2)\rho = \rho. \end{aligned}$$

Thus, $\lambda_1 w_1^* + \lambda_2 w_2^* \in W^*$, and so $W^*$ is convex. ∎

Theorem I.2 has the consequence that, if there are two distinct best approximations out of $W$ to $v$, there are infinitely many (in fact, uncountably many) best approximations.

A final general result gives a criterion that insures that, if there is a best approximation, there is only one. The normed linear space $V$ is said to have a

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But $(t - s)^2 = t^2 - 2st + s^2$; hence

$$B_m((t - s)^2; s) = B_m(t^2; s) - 2sB_m(t; s) + s^2B_m(1; s) = \frac{s(1 - s)}{m}$$

in view of (1.1.6), (1.1.7), and (1.1.8).

For $0 \le s \le 1$,

$$0 \le s(1 - s) \le \tfrac{1}{4};$$

hence, (1.1.12) implies that for $0 \le s \le 1$

$$|h(s) - B_m(h; s)| \le \varepsilon_1 + \frac{M}{2\delta^2 m}. \tag{1.1.13}$$

Now, if we choose $\varepsilon_1 = \varepsilon/2$, then (1.1.5) follows for any $m_0$ satisfying

$$m_0 > \frac{M}{\delta^2\varepsilon}.$$

In particular, then,

$$|g(t) - B_{m_0}(g; t)| < \varepsilon, \qquad 0 \le t \le 1,$$

and the theorem is proved by taking $p(x) = B_{m_0}(g; (x - a)/(b - a))$.

The Bernstein polynomials provide us with explicit approximations to a given continuous function. We have just seen that the sequence $\{B_n(h; t)\}$ converges uniformly to the continuous $h(t)$ on $[0, 1]$. The next question to consider is: How good an approximation can be obtained out of $P_n$? We first obtain a bound for the error

$$\max_{0 \le t \le 1} |h(t) - B_n(h; t)|.$$

To this end, we need some more detailed information about continuous functions. Let $f(x)$ be defined on $[a, b]$, the *modulus of continuity* of $f(x)$ on $[a, b]$, $\omega(\delta)$, is defined for $\delta > 0$ by

$$\omega(\delta) = \sup_{\substack{x_1, x_2 \in [a,b], \\ |x_1 - x_2| \le \delta}} |f(x_1) - f(x_2)|.$$

Note that the modulus of continuity depends on $\delta$, the function $f$, and the interval $[a, b]$, so that $\omega(\delta)$ is shorthand for $\omega(f; [a, b]; \delta)$. We need some properties of the modulus of continuity.

LEMMA 1.1. *If* $0 < \delta_1 \le \delta_2$, *then* $\omega(\delta_1) \le \omega(\delta_2)$.

LEMMA 1.2. $f(x)$ *is uniformly continuous on* $[a, b]$ *if and only if*

$$\lim_{\delta \to 0} \omega(\delta) = 0.$$

where $a_0, a_1, \ldots, a_n; b_1, \ldots, b_n$ are the Fourier coefficients of $g(\theta)$ defined in (1.1.17) and $\rho_{1,n}, \ldots, \rho_{n,n}$, $n = 1, 2, \ldots$ are any given real numbers.

LEMMA 1.4. *If $g(\theta)$ is continuous on $-\pi \leq \theta \leq \pi$ and has period $2\pi$, then*

$$q_n(g; \theta) = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\phi + \theta) u_n(\phi)\, d\phi, \tag{1.1.19}$$

*where*

$$u_n(\phi) = \frac{1}{2} + \sum_{k=1}^{n} \rho_{k,n} \cos k\phi. \tag{1.1.20}$$

*Proof.* If we substitute (1.1.17) into (1.1.18) and recall the identity $\cos A \cos B + \sin A \sin B \equiv \cos (A - B)$, we obtain

$$q_n(g; \theta) = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\phi) u_n(\phi - \theta)\, d\phi = \frac{1}{\pi} \int_{-\pi-\theta}^{\pi-\theta} g(\tau + \theta) u_n(\tau)\, d\tau.$$

But both $g$ and $u_n$ have period $2\pi$; hence

$$\frac{1}{\pi} \int_{-\pi-\theta}^{-\pi} g(\tau + \theta) u_n(\tau)\, d\tau = \frac{1}{\pi} \int_{\pi-\theta}^{\pi} g(\tau + \theta) u_n(\tau)\, d\tau.$$

Thus,

$$q_n(g; \theta) = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\tau + \theta) u_n(\tau)\, d\tau,$$

which was to be proved. ▌

LEMMA 1.5

$$|\theta| \leq \frac{\pi}{2} |\sin \theta| \qquad \text{for} \qquad 0 \leq |\theta| \leq \frac{\pi}{2}.$$

*Proof.* The second derivative of $-\sin \theta$ is positive for $0 < \theta \leq \pi/2$; hence $-\sin \theta$ is a convex function and the point $(\theta, -(2/\pi)\theta)$ of the chord joining $(0, 0)$ and $(\pi/2, -1)$ cannot be below the point $(\theta, -\sin \theta)$. Thus, $-(2/\pi)\theta \geq -\sin \theta$ for $0 \leq \theta \leq \pi/2$, and the lemma follows. ▌

LEMMA 1.6

$$\sin \theta \leq \theta, \qquad \theta \geq 0.$$

*Proof.* Let $k(\theta) = \theta - \sin \theta$. Then by the mean-value theorem there exists $\xi$, $0 \leq \xi \leq \theta$ such that $k(\theta) - k(0) = \theta k'(\xi)$ or $\theta - \sin \theta = \theta(1 - \cos \xi) \geq 0$. ▌

LEMMA 1.7. *Suppose $\rho_{1,n}, \ldots, \rho_{n,n}$ to be chosen in such a way that*

$$u_n(\phi) \geq 0, \qquad -\pi \leq \phi \leq \pi; \tag{1.1.21}$$

Suppose that $f(x) \in C[-1, 1]$. Then $g(\theta) = f(\cos\theta)$ is continuous on $0 \le \theta \le \pi$, and we define it on $-\pi \le \theta < 0$ by $g(\theta) = g(-\theta)$ to obtain an *even* function, continuous on $[-\pi, \pi]$ having period $2\pi$. Since $g(\theta)$ is *even*, it has a best uniform approximation by trigonometric polynomials of degree at most $n$ which is also even (cf. Exercise 1.1), call it $q_n^*$. An *even* trigonometric polynomial of degree $n$ has the form

$$q_n^*(\theta) = \frac{a_0}{2} + a_1 \cos\theta + \cdots + a_n \cos n\theta.$$

That is, all the sine terms have zero coefficients. It is easy to verify (cf. Exercise 1.6) that $\cos k\theta$ is a polynomial in $\cos\theta$ of degree at most $k$; hence,

$$q_n^*(\theta) = d_0 + d_1 \cos\theta + d_2(\cos\theta)^2 + \cdots + d_n(\cos\theta)^n,$$

and, if we put

$$p_n^*(x) = d_0 + d_1 x + \cdots + d_n x^n,$$

then according to (1.1.26) and Exercise 1.4

$$\max_{-1 \le x \le 1} |f(x) - p_n^*(x)| \le 6\omega\left(g; [-\pi, \pi]; \frac{1}{n}\right) \le 6\omega\left(f; [-1, 1]; \frac{1}{n}\right).$$

Finally, then, we have established

THEOREM 1.4[(2)] (JACKSON'S THEOREM). *If $f \in C[-1, 1]$, then*

$$E_n(f; [-1, 1]) \le 6\omega\left(\frac{1}{n}\right). \qquad (1.1.28)$$

This is our main result, and we turn at once to its immediate consequences.

COROLLARY 1.4.1. *If $f \in C[a, b]$, then*

$$E_n(f; [a, b]) \le 6\omega\left(\frac{b-a}{2n}\right).$$

*Proof.* Apply Exercise 1.7 to the result of Theorem 1.4.

COROLLARY 1.4.2. *If $f \in \mathrm{lip}_K\alpha$ on $[-1, 1]$, then*

$$E_n(f; [-1, 1]) \le 6Kn^{-\alpha}.$$

COROLLARY 1.4.3. *If $|f'(x)| \le M$ for $-1 \le x \le 1$, then*

$$E_n(f; [-1, 1]) \le 6Mn^{-1}.$$

*Proof.* See Exercise 1.8.

Theorem 1.6 is just a foreshadowing of the true state of affairs, however. As we shall show next, the curve $y = e(x)$ must touch the lines $y = \pm E_n(f)$ alternately at least $n + 2$ times, and this property characterizes the best uniform approximation of a continuous function by a polynomial of degree at most $n$. A set of $k + 1$ distinct points $x_0, \ldots, x_k$, satisfying $a \leq x_0 < x_1 < \cdots < x_{k-1} < x_k \leq b$ is called an *alternating set* for the error function $f - p_n$ if

$$|f(x_j) - p_n(x_j)| = \|f - p_n\|, \qquad j = 0, \ldots, k \tag{1.2.4}$$

and

$$[f(x_j) - p_n(x_j)] = -[f(x_{j+1}) - p_n(x_{j+1})], \qquad j = 0, \ldots, k - 1. \tag{1.2.5}$$

THEOREM 1.7. *Suppose $f \in C[a, b]$; $p_n^*$ is a best uniform approximation on $[a, b]$ to $f$ out of $P_n$ if and only if there exists an alternating set for $f - p_n^*$ consisting of $n + 2$ points.*

*Proof.* (i) Suppose $x_0, \ldots, x_{n+1}$ form an alternating set for $f - p_n^*$. We show that $p_n^*$ is a best approximation. If it is not, then there exists $q_n \in P_n$ such that

$$\|f - q_n\| < \|f - p_n^*\|. \tag{1.2.6}$$

Hence, in particular, since $x_0, \ldots, x_{n+1}$ form an alternating set,

$$|f(x_j) - q_n(x_j)| < \|f - p_n^*\| = |f(x_j) - p_n^*(x_j)|, \qquad j = 0, \ldots, n + 1. \tag{1.2.7}$$

(1.2.7) and (1.2.5) imply that the difference

$$[f(x_j) - p_n^*(x_j)] - [f(x_j) - q_n(x_j)] = q_n(x_j) - p_n^*(x_j)$$

alternates in sign as $j$ runs from 0 to $n + 1$. Thus the polynomial $q_n(x) - p_n^*(x) \in P_n$ has a zero in each interval $(x_j, x_{j+1})$, $j = 0, \ldots, n$, for a total of $n + 1$ zeros, which implies $q_n = p_n^*$. This contradicts (1.2.6), hence implies that $p_n^*$ is a best approximation and concludes the easier half of our proof.

(ii) Suppose that $p_n^*$ is a best approximation to $f$ and $f \notin P_n$. (If $f \in P_n$, the whole question is trivial.) Let a largest alternating set for $f - p_n^*$ consist of the $k + 1$ points $x_0, \ldots, x_k$ satisfying $a \leq x_0 < x_1 < \cdots < x_{k-1} < x_k \leq b$. In view of Theorem 1.6, $k \geq 1$. We wish to prove that $k \geq n + 1$. Suppose, then, that $k \leq n$, and let us put

$$\|f - p_n^*\| = \rho \qquad (>0).$$

Let $t_0, \ldots, t_s$ be points of $[a, b]$ chosen so that $a = t_0 < t_1 < \cdots < t_s = b$ and so that $e(x) = f(x) - p_n^*(x)$ satisfies

$$|e(\xi) - e(\eta)| < \tfrac{1}{2}\rho \tag{1.2.8}$$

Integrating both sides of (1.2.14) yields

$$\text{arc cos}\left(\frac{e(x)}{\rho}\right) = (n + 1)\theta + c,$$

where $x = \cos\theta$, $x \in [-1, x_1]$, $0 \leq \theta_1 \leq \theta \leq \pi$. Thus we obtain

$$e(x) = \rho \cos [(n + 1)\theta + c].$$

Now $e(-1) = -\rho$ since we are assuming that $e'(-1) \geq 0$; thus, $\cos [(n + 1)\pi + c] = -1$ and $c = m\pi$, where $m + n + 1$ is odd. Hence,

$$e(x) = \pm \rho \cos (n + 1)\theta, \tag{1.2.15}$$

$\cos (n + 1)\theta$ is a polynomial of degree $n + 1$ in $x = \cos\theta$; that is, $\cos (n + 1)\theta \in P_{n+1}$, and its leading coefficient is $2^n$ (cf. Exercise 1.6). (1.2.15) and (1.2.11) now imply that

$$e(x) = 2^{-n} \cos (n + 1)\theta. \tag{1.2.16}$$

[It is clear that if $e'(x) \leq 0$ in $[-1, x_1]$, we choose the negative square root on the right-hand side of (1.2.14), and the ensuing argument produces (1.2.16) again.]

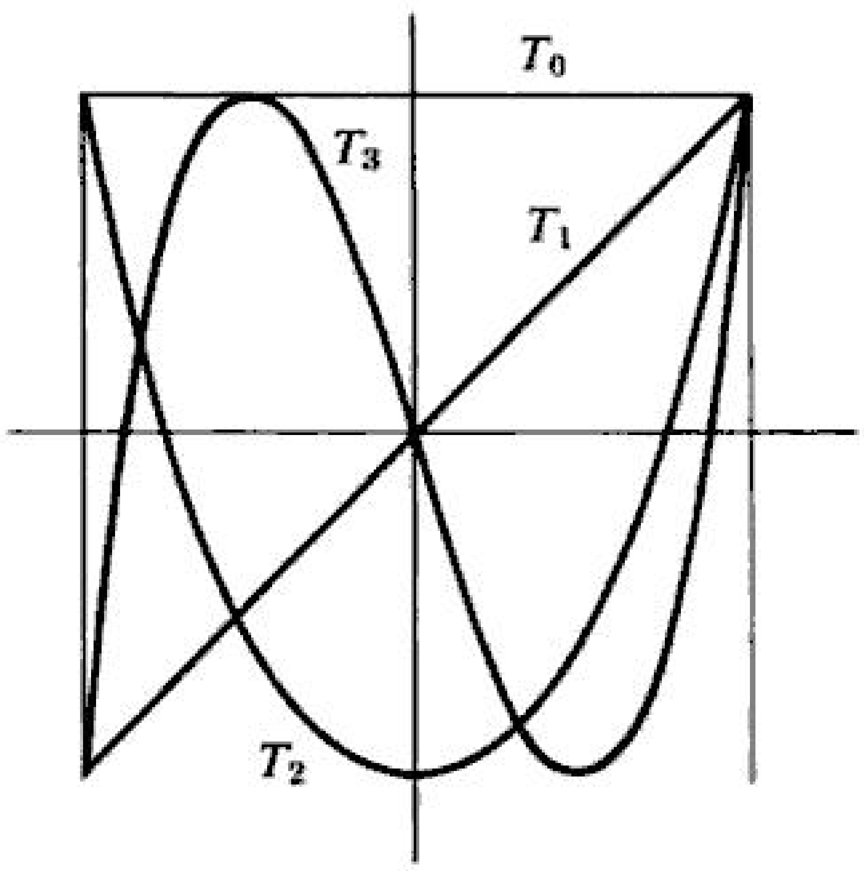


FIGURE 1.2

The polynomial $\cos k\theta$, where $x = \cos\theta$, $0 \leq \theta \leq \pi$, is called the *Chebyshev polynomial* of degree $k$ and we write

$$T_k(x) = \cos k\theta, \qquad k = 0, 1, 2, \ldots,\dagger \tag{1.2.17}$$

† The notation follows another transliteration from the Russian of "Chebyshev," one beginning with a T.

*for any* $p \in P_n$, $p \neq p_n^*$. *Moreover, for any* $X_{n+2} \subset [a, b]$,

$$\begin{aligned} E_n(f; X_{n+2}) &= \min_{p \in P_n} \max_{x \in X_{n+2}} |f(x) - p(x)| \\ &= \max_{x \in X_{n+2}} |f(x) - p_n^*(X_{n+2}; x)| \leq E_n(f; [a, b]) = E_n(f; X_{n+2}^*) \end{aligned} \tag{1.3.1}$$

*with equality possible in* (1.3.1) *only if* $p_n^*(X_{n+2}) = p_n^*$.

*Proof.* Let $X_{n+2}^*$ be an alternating set for $f - p_n^*$, then, in view of Theorems 1.11 and 1.12 (with $X_m = X_{n+2}^*$), $p_n^*$ is the best approximation to $f$ on $X_{n+2}^*$, and the first part of the theorem is established.

Suppose $p_n^*(X_{n+2}) \neq p_n^*$; then, by uniqueness (Theorem 1.12),

$$E_n(f; X_{n+2}) < \max_{x \in X_{n+2}} |f(x) - p_n^*(x)| \leq E_n(f; [a, b]).$$

Equality can hold in (1.3.1) if $p_n^*(X_{n+2}) = p_n^*$, and $X_{n+2}$ is an alternating set for $f - p_n^*$. ∎

We obtain, similarly, by replacing $[a, b]$ by $X_m$,

THEOREM 1.14. *If* $p_n^*(X_m) \in P_n$ *is the best approximation to* $f$ *on* $X_m$, *there exists* $X_{n+2}^* \subseteq X_m$ *such that*

$$\begin{aligned} E_n(f; X_m) = E_n(f; X_{n+2}^*) &= \max_{x \in X_{n+2}^*} |f(x) - p_n^*(X_m; x)| \\ &< \max_{x \in X_{n+2}^*} |f(x) - p(x)| \end{aligned} \tag{1.3.2}$$

*for any* $p \in P_n$, $p \neq p_n^*(X_m)$. *Moreover, for any* $X_{n+2} \subseteq X_m$,

$$E_n(f; X_{n+2}) \leq E_n(f; X_{n+2}^*) \tag{1.3.3}$$

*with equality possible only if* $p_n^*(X_{n+2}) = p_n^*(X_m)$.

Theorems 1.13 and 1.14 enable us to reduce the search for a best approximation out of $P_n$ to a set of $n + 2$ points. In the case of Theorem 1.13, this is not too helpful since there are infinitely many sets of $n + 2$ points in $[a, b]$. But in the case of Theorem 1.14 we see that, if we denote by $X_{m,i}$, $i = 1, \ldots, \binom{m}{n+2}$, the *finite* number of different subsets of $n + 2$ points of $X_m$, and find

$$\max_{i=1,\ldots,\binom{m}{n+2}} E_n(f; X_{m,i}) = E_n(f; X_{m,i^*}),$$

then

$$p_n^*(X_m) = p_n^*(X_{m,i^*}).$$

This procedure presupposes our ability to find $E_n(f; X_{n+2})$ and $p_n^*(X_{n+2})$ for any set of $n + 2$ distinct points, $X_{n+2}$. Our next task is to see how this can be done.

for some $\eta \in I$. But $p'(\xi) = 0$ since $p(x)$ has a relative extremum at $\xi$; hence

$$\|p\| = |p(\xi)| \le K + \frac{\delta_m^2}{2} |p''(\eta)|. \tag{1.3.18}$$

We need an upper bound for $|p''(\eta)|$. This is provided by the result of V. Markov (1.2.22) with $k = 2$:

$$|p''(\eta)| \le \frac{n^2(n^2 - 1)}{3} \|p\|. \tag{1.3.19}$$

Substituting (1.3.19) into (1.3.18) yields

$$\|p\| \le K + \frac{\delta_m^2}{6} n^2(n^2 - 1)\|p\| = K + \tau\|p\|,$$

from which (1.3.16) follows. ▮

LEMMA 1.9. *For $p \in P_n$*

$$\omega(p; I; \delta) \le \delta n^2 \|p\|. \tag{1.3.20}$$

*Proof.* If $x', x'' \in I$, then, by the mean-value theorem,

$$p(x') - p(x'') = (x' - x'')p'(\eta).$$

But, relying on Markov's result (1.2.22),

$$|p'(\eta)| \le n^2 \|p\|$$

and, hence,

$$|p(x') - p(x'')| \le \delta n^2 \|p\|,$$

which implies (1.3.20). ▮

THEOREM 1.16. *If $f(x)$ is continuous on $I$, then*

$$E_n(f; I) - \omega(f; I; \delta_m) - \delta_m n^2 \left[\frac{\|f\| + E_n(f; I)}{1 - \tau}\right] \le E_n(f; X_m) \le E_n(f; I), \tag{1.3.21}$$

*provided that $\delta_m$ satisfies* (1.3.15).

*Proof.* Let $p_n^*$ be the best approximation to $f$ on $I$ and $q_n^*$ the best approximation to $f$ on $X_m$. Then

$$E_n(f; X_m) \le \max_{x \in X_m} |f(x) - p_n^*(x)| \le \max_{x \in I} |f(x) - p_n^*(x)| = E_n(f; I), \tag{1.3.22}$$

and the right-hand inequality in (1.3.21) is proved.

If $M_\mu > \rho_\mu$, we repeat the exchange procedure to obtain a new reference $\{x_\alpha\}$ such that $\rho_\alpha > \rho_\mu$. At each step we get a new reference, and since $X_m$ contains only finitely many references, the process terminates in a finite number of steps. Thus the exchange method is, in general, a more sensible strategy than proceeding through all the references $X_{n+2,i}$ of $X_m$ in a random fashion since the reference deviations increase monotonically.

The question of which reference to choose at the start arises. A practical answer can be given by jumping ahead to our chapter on least-squares approximation (Chapter 2). If $q_n$ is the best least-squares approximation of degree at most $n$ to $f$ on $X_m$, that is,

$$\sum_{x \in X_m} [f(x) - q_n(x)]^2 < \sum_{x \in X_m} [f(x) - p(x)]^2$$

for any $p \in P_n$, $p \neq q_n$, then as we shall see, there exist $n + 2$ points of $X_m$ on which $f - q_n$ alternates in sign. These points are a good choice for a starting reference. (It will also turn out that $q_n$ can always be found quite easily, so we have not just exchanged one difficulty for another of equal magnitude.) Let us examine the example we worked out on p. 36, that is, approximation by quadratics to $|x|$ on $X_5$. The least-squares approximation on $X_5$ turns out to be $q_2 = \frac{6}{7}x^2 + \frac{6}{35}$, and $|x| - q_2$ alternates in sign on $X_{5,1}$ and $X_{5,5}$, so that we are led at once to the optimal references. Suppose, however, that we start with $X_{5,2} = \{-1, -\frac{1}{2}, 0, 1\}$ as a reference, $\sigma$. Then $p_\sigma = \frac{8}{9}x^2 - \frac{1}{9}x + \frac{1}{9}$, $M_\sigma = \frac{2}{9}$, and $x_j = \frac{1}{2}$. $|x| - p_\sigma = -\frac{1}{9}$ at $x = 0$, $\frac{2}{9}$ at $x = \frac{1}{2}$, and $\frac{1}{9}$ at $x = 1$. We therefore exchange the point 1 of the reference for the point $\frac{1}{2}$, obtaining a new reference, namely $X_{5,1}$, which we know to be optimal. The point is that we were not led to $X_{5,3}$. For this and other variants of the exchange method, we refer the reader to the following literature: Rice [1], Meinardus [1], Remez [1], Stiefel [1].

### 1.4.2 Linear Programming

Another approach to the problem of finding best approximations on $X_m$ is to express the problem as a linear programming problem. In addition to the $n + 1$ unknown coefficients of $p_n^*(X_m)$, we introduce a new variable, $\rho$, as follows. The condition

$$\max_{x \in X_m} |f(x) - p(x)| = \rho$$

can be written

$$-\rho \leq f(x_j) - \sum_{i=0}^{n} a_i x_j^i \leq \rho, \qquad j = 1, \ldots, m.$$

where $t = a - (a^2 - 1)^{1/2}$. Use this result together with Exercise 1.9 to find the best approximation out of $P_n$ on $[-1, 1]$ to $q(x)/(x - a)$ for any $q \in P_{n+1}$.

[*Hint:* (Rivlin [1]) With $x = \cos\theta$, it is possible to sum the series

$$\sum_{j=0}^{\infty} t^j T_j(x) = \sum_{j=0}^{\infty} t^j \cos j\theta = \operatorname{Re} \sum_{j=0}^{\infty} (t\, e^{i\theta})^j.$$

For the second part, find $p \in P_n$ such that

$$\frac{q(x)}{x-a} = \frac{c}{x-a} - p.]$$

1.21 Show that

$$T_n^{(k)}(x) > 0 \qquad \text{for} \qquad x \geq 1, \qquad k = 0, \ldots, n$$

and

$$\operatorname{sgn} T_n^{(k)}(x) = (-1)^n \qquad \text{for} \qquad x \leq -1, \qquad k = 0, \ldots, n.$$

[*Hint:* Use Rolle's Theorem.]

1.22 Determine the conditions under which there can be equality in (1.2.19), (Theorem 1.10).

1.23 Verify that (1.3.10) is equivalent to

$$\lambda = \sum_{i=1}^{n+2} (-1)^i \lambda_i f(x_i),$$

where

$$\lambda_i = \frac{1/|\omega'(x_i)|}{\sum_{i=1}^{n+2} (1/|\omega'(x_i)|)}, \qquad i = 1, \ldots, n+2,$$

and hence

$$\lambda_i > 0 \qquad \text{and} \qquad \sum_{i=1}^{n+2} \lambda_i = 1.$$

1.24 Show that, in the notation of Exercise 1.23,

$$\sum_{i=1}^{n+2} (-1)^i \lambda_i q(x_i) = 0$$

for any $q \in P_n$.

1.25 (L. Smith) Given $f \in C[-1, 1]$, consider the problem of finding $\bar{p} \in P_n$ with the property that

$$f(x) \geq \bar{p}(x), \qquad -1 \leq x \leq 1,$$

and

$$\max_{-1 \leq x \leq 1} [f(x) - \bar{p}(x)] \leq \max_{-1 \leq x \leq 1} [f(x) - p(x)]$$

a system of $(n + 1)$ linear equations for the $(n + 1)$ unknown coefficients of $q_n^*(x)$. If we write

$$q_n^*(x) = \xi_0 + \xi_1 x + \cdots + \xi_n x^n,$$

the system (2.1.4) may be written

$$\sum_{j=0}^{n} a_{ij}\xi_j = b_i, \qquad i = 0, \ldots, n, \tag{2.1.5}$$

where

$$a_{ij} = \int_{-1}^{1} x^{i+j} w(x)\, dx \tag{2.1.6}$$

and

$$b_i = \int_{-1}^{1} x^i f(x) w(x)\, dx. \tag{2.1.7}$$

In principle, then, we can determine $\xi_0, \ldots, \xi_n$ from (2.1.5), (which are called the "normal" equations) and thereby obtain $q_n^*$ explicitly. However, when $n$ is at all large, say $n \geq 7$, there appear, in the simple case that $w(x) \equiv 1$, formidable *numerical* difficulties in solving the normal equations. (For a discussion of the reason for these difficulties, see Forsythe [1].) It is possible to avoid these computational difficulties and find $q_n^*$ in an extremely simple manner by observing that $\{1, x, x^2, \ldots, x^n\}$ is not the only set of functions that spans $P_n$, the space of polynomials of degree at most $n$. That is to say, if $p_0, p_1, \ldots, p_n \in P_n$ are linearly independent, then every $p \in P_n$ has a unique expression of the form

$$p = \alpha_0 p_0 + \alpha_1 p_1 + \cdots + \alpha_n p_n. \tag{2.1.8}$$

We are going to determine a set $\{p_0, p_1, \ldots, p_n\}$ which will be *orthogonal* with respect to the given weight function, $w(x)$. That is, we seek $p_0, p_1, \ldots, p_n \in P_n$ such that

$$\int_{-1}^{1} p_j(x) p_k(x) w(x)\, dx = 0, \qquad j \neq k, \qquad j, k = 0, \ldots, n. \tag{2.1.9}$$

If, in addition to (2.1.9), we also have

$$\int_{-1}^{1} p_j^2(x) w(x)\, dx = 1, \qquad j = 0, \ldots, n, \tag{2.1.10}$$

then $\{p_0, p_1, \ldots, p_n\}$ is called a set of *orthonormal* polynomials with respect to $w(x)$. Before proceeding to construct such an orthonormal set, let us see how it simplifies the least-squares approximation problem. Suppose $p_0, \ldots, p_n \in P_n$ satisfy (2.1.9) and (2.1.10). Then $p_0, p_1, \ldots, p_n$ are linearly independent (cf. Exercise 2.1). Let

$$q_n^*(x) = \lambda_0 p_0(x) + \lambda_1 p_1(x) + \cdots + \lambda_n p_n(x). \tag{2.1.11}$$

Therefore, the Legendre polynomials satisfy the recurrence relationship

$$P_{n+1}(x) = \binom{2n+2}{n+1} 2^{-(n+1)} x\tilde{P}_n(x) - \binom{2n+2}{n+1} 2^{-(n+1)} \frac{n^2}{4n^2-1} \tilde{P}_{n-1}(x)$$

$$= \frac{2n+1}{n+1} xP_n(x) - \frac{n}{n+1} P_{n-1}(x)$$

or

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x). \tag{2.2.6}$$

The first few Legendre polynomials are thus seen to be

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \tfrac{3}{2}x^2 - \tfrac{1}{2}, \quad P_3(x) = \tfrac{5}{2}x^3 - \tfrac{3}{2}x,$$
$$P_4(x) = \tfrac{35}{8}x^4 - \tfrac{15}{4}x^2 + \tfrac{3}{8}.$$

They are depicted in Figure 2.1.

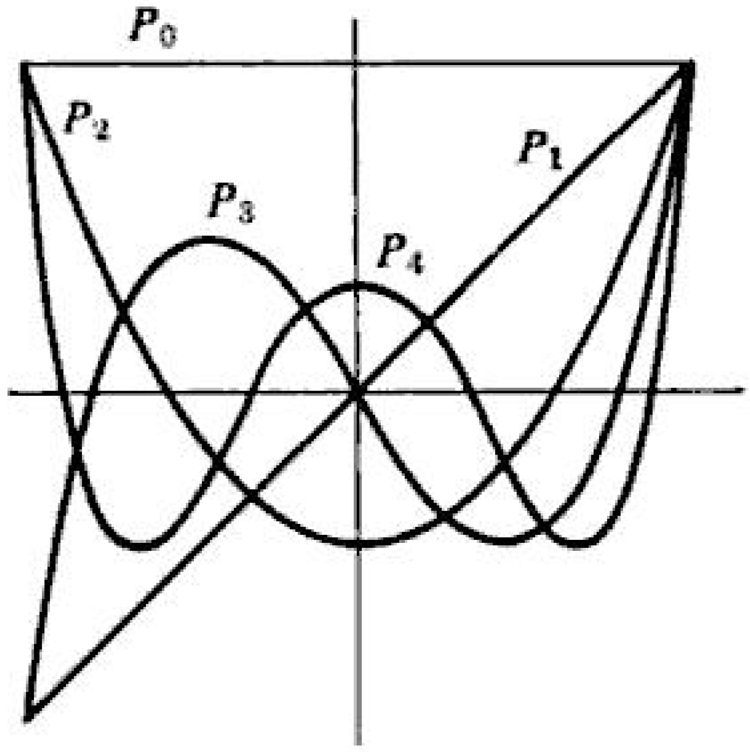


FIGURE 2.1

Next let us consider the Jacobi polynomials in the case $\alpha = \beta = -\frac{1}{2}$, so that

$$w(x) = (1 - x^2)^{-1/2}. \tag{2.2.7}$$

We saw [Exercise 1.16(c)] that the Chebyshev polynomials defined as

$$T_n(x) = \cos n\theta, \qquad n = 0, 1, 2, \ldots, \tag{2.2.8}$$

where $x = \cos\theta$, $0 \le \theta \le \pi$, are orthogonal with respect to the weight function given in (2.2.7). Thus, in view of (2.2.1),

$$P_j^{(-1/2,-1/2)} = \frac{1\cdot 3\cdot 5\cdots(2j-1)}{2^j j!} T_j(x).$$

(see p. 17). Since $g$ is an even function, $b_k = 0$, $k = 1, \ldots, n$, and

$$s_n(g;\theta) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos k\theta = \frac{a_0}{2}\sum_{k=1}^{n} a_k T_k(x). \qquad (2.4.2)$$

The definition of the $a_k$ [(1.1.17), p. 17] reveals that (2.4.2) is the least-squares approximation of degree $n$ with respect to (2.4.1).

Now, according to Lemma 1.4, if $g(\theta)$ is any continuous function on $-\pi \le \theta \le \pi$ of period $2\pi$,

$$s_n(g;\theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\phi + \theta) v_n(\phi)\, d\phi,$$

where

$$v_n(\phi) = \frac{1}{2} + \sum_{k=1}^{n} \cos k\phi.$$

The reader should have no difficulty in verifying that

$$v_n(\phi) = \frac{\sin\{(2n+1)/2\}\phi}{2\sin(\phi/2)}, \qquad (\phi \ne 2m\pi).$$

Thus,

$$\begin{aligned} s_n(g;\theta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\phi+\theta)\frac{\sin\{(2n+1)/2\}\phi}{\sin(\phi/2)}\, d\phi \\ &= \frac{1}{2\pi}\int_{0}^{\pi} [g(\theta+\phi) + g(\theta-\phi)]\frac{\sin\{(2n+1)/2\}\phi}{\sin(\phi/2)}\, d\phi. \end{aligned}$$

We have therefore established

LEMMA 2.1. *If $g(\theta)$ is continuous on $-\pi \le \theta \le \pi$, has period $2\pi$, and satisfies*

$$|g(\theta)| \le M, \qquad -\pi \le \theta \le \pi,$$

*then its Fourier partial sums satisfy*

$$\max_{-\pi \le \theta \le \pi} |s_n(g;\theta)| \le M L_n, \qquad (2.4.3)$$

*where*

$$L_n = \frac{1}{\pi}\int_0^{\pi} \frac{|\sin(n+\frac{1}{2})\phi|}{\sin(\phi/2)}\, d\phi. \qquad (2.4.4)$$

The numbers $L_n$ are known as *Lebesgue's constants.*

*Remark.* The (even) function

$$g(\theta) = \operatorname{sgn} \frac{\sin(n+\frac{1}{2})\theta}{\sin(\theta/2)} \qquad (2.4.5)$$

2.4 Prove that $f - q_n^*$ changes sign at $n + 1$ points of $(-1, 1)$ if $f \notin P_n$. (See p. 42 for an application of this fact.)

[*Hint:* If $f - q_n^*$ changes sign at at most $r \leq n$ points of $(-1, 1)$, there exists $p \in P_n$ such that $(f - q_n^*)p \geq 0$ throughout $I$. Show that this contradicts (2.1.2) since $(f - q_n^*)p \in C(I)$ and $w(x) > 0$ except for a finite number of points.]

2.5 Show that the least-squares approximation of degree $n - 1$ to $f(x) = x^n$ is $q_{n-1}^* = x^n - \tilde{p}_n$, where $\tilde{p}_n$ is the orthogonal polynomial of degree $n$ determined by $w(x)$ and (2.1.16).

2.6 Show that the orthogonal polynomial $\tilde{p}_n$ (and hence $p_n$) has $n$ distinct simple zeros in $(-1, 1)$.

[*Hint:* Use Exercises 2.4 and 2.5.]

2.7 With the notation of Exercise 2.2, show that

$$\lim_{n \to \infty} S_n(f; w) = 0.$$

[*Hint:* $S_n(f; w) \leq \|f - p_n^*\|_2^2$, where $p_n^*$ is the best *uniform* approximation to $f$ on $I$.]

2.8 Show that another form of (2.1.18) is

$$\beta_k = \frac{(\tilde{p}_k, \tilde{p}_k)}{(\tilde{p}_{k-1}, \tilde{p}_{k-1})}. \tag{2.4.11}$$

[*Hint:* Replace $k$ by $k - 1$ in (2.1.16), then multiply both sides by $\tilde{p}_k w$ and integrate.]

2.9 Prove that the Jacobi polynomial $P_j^{(\alpha,\alpha)}$ is an odd function for odd $j$ and an even function for even $j$.

[*Hint:* Use mathematical induction, (2.1.16) and the fact that $w(x)$ is an even function in this case.]

2.10 Show that, if $w(x) \equiv 1$, the orthogonal polynomials defined by (2.1.16) satisfy

$$(\tilde{p}_n, \tilde{p}_n) = \frac{2}{2n + 1} \tilde{p}_n^2(1), \qquad n = 0, 1, \ldots. \tag{2.4.12}$$

[*Hint:* Integration by parts yields

$$(\tilde{p}_n, \tilde{p}_n) = \tilde{p}_n^2(1) + \tilde{p}_n^2(-1) - 2n(\tilde{p}_n, \tilde{p}_n).]$$

2.11 Show that

$$(P_n, P_n) = \frac{2}{2n+1}, \qquad n = 0, 1, 2, \ldots.$$

# CHAPTER 3

# LEAST-FIRST-POWER APPROXIMATION

This chapter is devoted to the study of polynomials that minimize the integral of the absolute deviation from a given function. After characterizing best approximations and proving uniqueness, some special cases in which the best approximation is easily attainable are discussed. There follows consideration of approximation on a finite point set, and the chapter closes with a brief examination of some computational aspects of the problem.

## *3.1 Approximation on an Interval*

In view of Theorem I.1, we know that, given $f(x)$ continuous on $I$: $[-1, 1]$, there exists a polynomial $r_n^* \in P_n$ such that

$$\|f - r_n^*\|_1 = \int_{-1}^{1} |f(x) - r_n^*(x)|\, dx \le \|f - p\|_1 \tag{3.1.1}$$

for all $p \in P_n$. In contrast to the case of least-squares approximation, we cannot conclude that $r_n^*$ is unique from Theorem I.3, and must therefore leave aside, for the moment, the question of uniqueness. Any $r_n^*$ satisfying (3.1.1) we call a least-first-power approximation to $f$ out of $P_n$.

As is our custom, we shall begin by characterizing $r_n^*$. To this end we need some new notation. If $g \in C(I)$, let

$$Z(g) = \{x \in I/g(x) = 0\}; \tag{3.1.2}$$

that is, $Z(g)$ is the set of zeros of $g$. Clearly, $Z(g)$ is a closed set. If $x \in Z(g)$ is such that every open interval of the real line that contains $x$ also contains points that are *not* in $Z(g)$, then $x$ is a boundary point of $Z(g)$.

DEFINITION. We call the boundary points of $Z(g)$ *essential zeros* of $g$, and denote them by $Z'(g)$.

(3.1.9) and (3.1.10) imply that

$$0 = \int_{-1}^{1} \left\{ |f(x) - p_0(x)| - \frac{1}{2}|f(x) - p_1(x)| - \frac{1}{2}|f(x) - p_2(x)| \right\} dx. \tag{3.1.11}$$

Since

$$\begin{aligned} |f(x) - p_0(x)| &= \left| f(x) - \frac{p_1(x) + p_2(x)}{2} \right| \\ &= \frac{1}{2} |[f(x) - p_1(x)] + [f(x) - p_2(x)]| \\ &\le \frac{1}{2}|f(x) - p_1(x)| + \frac{1}{2}|f(x) - p_2(x)|, \end{aligned}$$

we learn that the integrand in (3.1.11) is nonpositive. It is also continuous and therefore must be identically zero. This means that, if $f - p_0$ has $k$ distinct zeros in $I$, $k \le n$. For suppose that $k > n$, and let the zeros of $f - p_0$ be $x_1, \ldots, x_k$. Then, for $i = 1, \ldots, k$,

$$\begin{aligned} 0 &= |f(x_i) - p_0(x_i)| - \tfrac{1}{2}|f(x_i) - p_1(x_i)| - \tfrac{1}{2}|f(x_i) - p_2(x_i)| \\ &= -\tfrac{1}{2}|f(x_i) - p_1(x_i)| - \tfrac{1}{2}|f(x_i) - p_2(x_i)|, \end{aligned}$$

from which we conclude that

$$f(x_i) - p_1(x_i) = f(x_i) - p_2(x_i) = 0.$$

Therefore, $p_1 - p_2 \in P_n$ has $k > n$ zeros, and so $p_1 = p_2$, which is contrary to our assumption.

Corollary 3.1.1 is now applicable with $r_n^* = p_0$ and tells us that

$$\int_{-1}^{1} \operatorname{sgn}[f(x) - p_0(x)] \cdot p(x)\, dx = 0 \tag{3.1.12}$$

for all $p \in P_n$. Let $t_1 < t_2 < \cdots < t_s$ be those zeros of $f - p_0$ located in $(-1, 1)$ at which $f - p_0$ changes sign. Then, certainly, $s \le k \le n$, and if we put $t_0 = -1$ and $t_{s+1} = 1$, (3.1.12) may be rewritten (possibly after being multiplied by $-1$) as

$$\sum_{j=0}^{s} (-1)^{s-j} \int_{t_j}^{t_{j+1}} p(x)\, dx = 0. \tag{3.1.13}$$

If we put $p(x) = (x - t_1) \cdots (x - t_s)$, then $p \in P_n$ and

$$\operatorname{sgn} \int_{t_j}^{t_{j+1}} p(x)\, dx = (-1)^{s-j}, \qquad j = 0, \ldots, s,$$

contradicting (3.1.13). The assumption that $p_1 \ne p_2$ has led to a contradiction. ▮

Now let $r$ be the smallest integer such that

$$2\sum_{i=1}^{r}|x_i| \geq \sum_{i=1}^{l}|x_i|;$$

then

$$\frac{f(x_r)}{x_r}x$$

is a least-first-power approximation of the required form. Discuss uniqueness.

[*Hint:* While Theorem 3.5 cannot be applied, since we are not admitting all polynomials of degree 1 as approximators, show that an exact analogue of Theorem 3.5 holds when the family of approximators consists of $\{ax\}$. Then apply this result. What is the result when the $w_i$ are arbitrary positive numbers?]

This problem has an interesting history. The result is the algebraic formulation by Laplace of a geometric method due to Boscovich. A fascinating account of Boscovich's approach and its subsequent ramifications as well as the history of the relative merits of least-first-power and least-squares fitting of data can be found in Eisenhart [1].

3.11 Given $X_3$, show that the least-first-power approximation by polynomials of degree at most 1, with $w_1 = w_2 = w_3 = 1$, to $f$ on $X_3$ is provided by the line passing through $(x_1, f(x_1))$ and $(x_3, f(x_3))$.

Suppose $m = n + 2, q_1, \ldots, q_s$ to be the extreme points of $B$ and $f \notin P_n$ in Exercises 3.12 through 3.15.

3.12 Show that there is exactly one point, call it $x_i \in X_m$, at which $q_i(x_i) \neq f(x_i)$, the points $x_1, \ldots, x_s$ are distinct, and each of $q_1, \ldots, q_s$ agrees with $f$ on $x_{s+1}, \ldots, x_{n+2}$.

3.13 If

$$p_0 = \lambda_1 q_1 + \cdots + \lambda_s q_s,$$

where $\lambda_j > 0, j = 1, \ldots, s$, and $\lambda_1 + \cdots + \lambda_s = 1$, show that $p_0$ agrees with $f$ precisely on $x_{s+1}, \ldots, x_{n+2}$.

3.14 If $p_0 \in B$ agrees with $f$ on $n + 1$ points, show that $p_0$ is an extreme point of $B$.

*Remark.* The extreme points of $B$ are now seen to be found among the (at most) $n + 2$ polynomials that agree with $f$ on some $n + 1$ points of $X_m$, and hence all of $B$ can be determined by direct examination of (at most) $n + 2$ polynomials.

3.15 Suppose that $q \in B$ is an extreme point of $B$ agreeing with $f$ on $x_1, \ldots, x_{n+1}$. Show that, for any $g \notin P_n$, the unique $p_0 \in P_n$ that agrees with $g$

*Proof.* Let $p^* \in P_k$ be the best (uniform) approximation to $f$ on $I$. Then

$$|f(x) - L_k(x)| \le |f(x) - p^*(x)| + |p^*(x) - L_k(x)|. \tag{4.1.6}$$

By the uniqueness of the Lagrange interpolating polynomial,

$$p^*(x) = L_k(p^*, X; x),$$

and hence

$$p^*(x) - L_k(f, X; x) = L_k(p^*, X; x) - L_k(f, X; x) = L_k(p^* - f, X; x).$$

(4.1.6) now yields

$$|f(x) - L_k(x)| \le E_k + L_k(p^* - f, X; x). \tag{4.1.7}$$

But

$$|L_k(p^* - f, X; x)| \le \max_{-1 \le x \le 1} |p^*(x) - f(x)| \cdot \max_{-1 \le x \le 1} \sum_{j=1}^{k+1} |l_j^{(k)}(x)|. \tag{4.1.8}$$

The theorem now follows from (4.1.7) and (4.1.8). ▮

The function

$$\lambda_k(X; x) = \sum_{j=1}^{k+1} |l_j^{(k)}(x)|, \qquad k = 0, 1, \ldots, \tag{4.1.9}$$

which appears in (4.1.5), is called the *Lebesgue function* of order $k$ of $X$. Note that it does not depend on $f$. The quantity

$$\Lambda_k(X) = \max_{-1 \le x \le 1} \lambda_k(X; x)$$

is called the *Lebesgue constant* of order $k$ of $X$. (4.1.5) may now be written concisely as

$$G_k \le E_k(1 + \Lambda_k), \qquad k = 0, 1, \ldots. \tag{4.1.10}$$

But according to Jackson's Theorem (Theorem 1.4), $E_k \le 6\omega(1/k)$ and, hence,

$$G_k \le 6(1 + \Lambda_k)\omega(1/k). \tag{4.1.11}$$

Thus, insofar as Theorem 4.1 is informative, it tells us that, given $X$ and $f \in C(I)$, the sequence of interpolating polynomials converges uniformly to $f$ on $I$ if $\Lambda_k\omega(1/k) \to 0$ as $k \to \infty$. If we know more about $f$, say that it has a certain number of derivatives, we can use the appropriate variant of Jackson's Theorem (see the discussion following Theorem 1.4) to bound $E_k$ in (4.1.10) and obtain results analogous to (4.1.11). We must still estimate the size of $\Lambda_k(X)$ and study the implications of these estimates. This we do in the next section.

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# An Introduction to the Approximation of Functions

## Theodore J. Rivlin

The needs of automatic digital computation have spurred an enormous revival of interest in methods of approximating continuous functions by functions which depend only on a finite number of parameters. This concise but wide-ranging text provides an introduction to some of the most significant of these methods, with particular emphasis on approximation by polynomials.

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