Ergebnisse der Mathematik und ihrer Grenzgebiete

Volume 40

3. Folge

A Series of Modern Surveys in Mathematics

Springer-Verlag Berlin Heidelberg GmbH

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Profinite Groups

Springer

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Library of Congress Cataloging-in-Publication Data applied for

Die Deutsche Bibliothek - CIP-Einheitsaufnahme
Ribes, Luis:
Profinite groups / Luis Ribes ; Pavel Zalesskii. - Berlin ; Heidelberg
; New York ; Barcelon ; Hong Kong ; London ; Milan ; Paris ; Singapore
; Tokyo : Springer, 2000
(Ergebnisse der Mathematik und ihrer Grenzgebiete ; Folge 3,Vol. 40)
ISBN 978-3-642-08632-8 ISBN 978-3-662-04097-3 (eBook)
DOI 10.1007/978-3-662-04097-3

Mathematics Subject Classification (1991):
20E18, 20J05, 12G05, 20E05, 20E06

ISSN 0071-1136
ISBN 978-3-642-08632-8

Originally published by Springer-Verlag Berlin Heidelberg New York in 2000
Softcover reprint of the hardcover 1st edition 2000

Typesetting: Camera-ready copy produced by the authors' output file
using a Springer TEX macro package
Printed on acid-free paper SPIN 10479976 44/3143/LK - 5 4 3 2 1 0

Para Marisa, Alfonso y David

Людмиле и Алику

Preface

The aim of this book is to serve both as an introduction to profinite groups and as a reference for specialists in some areas of the theory. In neither of these two aspects have we tried to be encyclopedic. After some necessary background, we thoroughly develop the basic properties of profinite groups and introduce the main tools of the subject in algebra, topology and homology. Later we concentrate on some topics that we present in detail, including recent developments in those areas.

Interest in profinite groups arose first in the study of the Galois groups of infinite Galois extensions of fields. Indeed, profinite groups are precisely Galois groups and many of the applications of profinite groups are related to number theory. Galois groups carry with them a natural topology, the Krull topology. Under this topology they are Hausdorff compact and totally disconnected topological groups; these properties characterize profinite groups. Another important fact about profinite groups is that they are determined by their finite images under continuous homomorphisms: a profinite group is the inverse limit of its finite images. This explains the connection with abstract groups. If G is an infinite abstract group, one is interested in deducing properties of G from corresponding properties of its finite homomorphic images. The kernels of all homomorphisms of G into finite groups form a fundamental system of neighborhoods for a topology on G, and completion of G with respect to this topology gives a profinite group. In the last decades there has been an extensive literature on profinite groups and one of the aims of this book is to present some of these important results.

The first comprehensive exposition of the theory of profinite groups appeared in the book 'Cohomologie Galoisienne' by J-P. Serre in 1964. Its emphasis is on cohomological properties and their applications to field theory and number theory. This deceptively slim volume contains a wealth of information, some of it not found elsewhere. We have learnt a great deal from Serre's book throughout the years and this, no doubt, is reflected in our exposition in the present book.

We describe briefly the contents of our book. The first three chapters deal with the basic tools and the main properties of profinite groups. In Chapter 1 we have collected information about inverse and direct limits and their algebraic and topological properties, which is used throughout the book. Chapter

2 contains a fairly detailed account of general profinite groups. The results are presented in the context of pro-$\mathcal{C}$ groups (inverse limits of groups in $\mathcal{C}$), where $\mathcal{C}$ is a convenient class of finite groups, which includes the classes of profinite and pro-p groups as particular cases. The minimum we require of such a class $\mathcal{C}$ is that it should be a 'formation' (i.e., closed under taking quotients and finite subdirect products); but often we assume that $\mathcal{C}$ is a 'variety' (i.e., closed under taking subgroups, quotients and finite direct products). Although this approach requires the reader to become familiar with a little more terminology (but not much more than what is indicated above), this is compensated by being able to bring many related concepts and results together. Sometimes we assume throughout a chapter or a section that $\mathcal{C}$ satifies certain conditions; when that happens we indicate those assumptions in italics at the beginning of the chapter or section.

The main properties of free profinite (pro-$\mathcal{C}$) groups are developed in Chapter 3. These includes several useful characterizations in terms of lifting maps à la Iwasawa and the study of the structure of open subgroups of free pro-$\mathcal{C}$ groups. Chapter 4 considers properties of particular profinite groups, including profinite abelian groups, Frobenius profinite groups and automorphism groups of finitely generated profinite groups.

Chapters 5-7 deal with homological aspects of profinite groups. In Chapter 5, we consider modules over profinite rings, particularly complete group rings, and constructions involving them. Chapter 6 establishes the fundamental results of homology and cohomology groups of profinite groups. Here we combine a computational approach with a conceptual one: on the one hand, we define homology and cohomology groups by means of standard resolutions, and on the other hand, we give a more abstract description, using the language of universal functors. Chapter 7 contains cohomological characterizations of projective profinite groups and the Tate characterization of free pro-p groups.

Chapter 8 considers closed normal subgroups of free profinite groups, and in particular, conditions under which such subgroups are free profinite. We also study similar properties for closed subnormal subgroups and accessible subgroups. This chapter includes Mel'nikov's theory of homogeneous groups, which gives a description of certain closed subgroups of free pro-$\mathcal{C}$ groups (other than pro-p).

Chapter 9 establishes the main properties of the basic 'free constructions' of profinite groups: free and amalgamated products and HNN-extensions. This is the beginning of the theory of profinite groups acting on 'profinite trees', which we shall develop in a subsequent book.

The last section of each chapter gives some of the history of the theory that has been developed, and indicates the names of the main contributors. These sections also include statements or references to results not treated in the main body of the chapters.

Throughout the text we have included a series of open questions that are also gathered at the end of book.

We thank Hendrik Lenstra Jr. for his suggestion that a book such as this should be written for the Ergebnisse Series. His contagious optimism and enthusiasm, and his interest in our ideas and projects have been very uplifting and helpful.

Several colleagues and friends have read parts of the book. We are specially grateful to Zoé Chatzidakis, Juan Ramón Delgado, John Dixon, Otto Kegel and Wolfgang Herfort for their comments and corrections; the errors and misprints that may remain are attributable entirely to us. We are greatly indebted to Jean-Pierre Serre for sharing with us some of his ideas and for his help in Section 6.9.

Part of this book was written while one of us (LR) was on sabbatical at the UNED in Madrid at the invitation of Emilio Bujalance. The congenial mathematical atmosphere that our colleagues have created there was very conducive to our work. It is a pleasure to thank them for wonderful discussions (mathematical and otherwise) and for their friendship. The advice of Javier Pérez regarding xy-pic was very useful and we thank him for the time he spent teaching us the tricks.

In the Summer of 1988 both authors participated in the program Research in Pairs of the Mathematisches Forschunginstitut in Oberwolfach while writing this book; we thank the Mathematisches Forschunginstitut for the use of the excellent Library there and for the opportunity to work together and uninterrupted in such quiet and confortable quarters in the beautiful and relaxing Schwarzwald.

The first author gratefully acknowledges the support of the National Science and Engineering Research Council of Canada and the Dirección General de Investigación y Desarrollo of Spain.

The second author thanks the Austrian Science Foundation and Fundação de Apoio à Pesquisa do Distrito Federal (Brazil) for support.

Responsability for the writing of this book: L. Ribes has written most of the material in Chapters 1-8; the main exceptions are Section 4.5 and parts of Sections 4.4, 4.7, 5.6 and 8.3 which were written by P. Zalesskii; translation from Russian done by P. Zalesskii was important in the writing of Sections 8.5 and 8.10. Chapter 9 has been written by both authors. The editorial work for the final version of the book has been done by both authors.

January, 2000

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