WAVELET ANALYSIS

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Howard L. Resnikoff
Raymond O. Wells, Jr.

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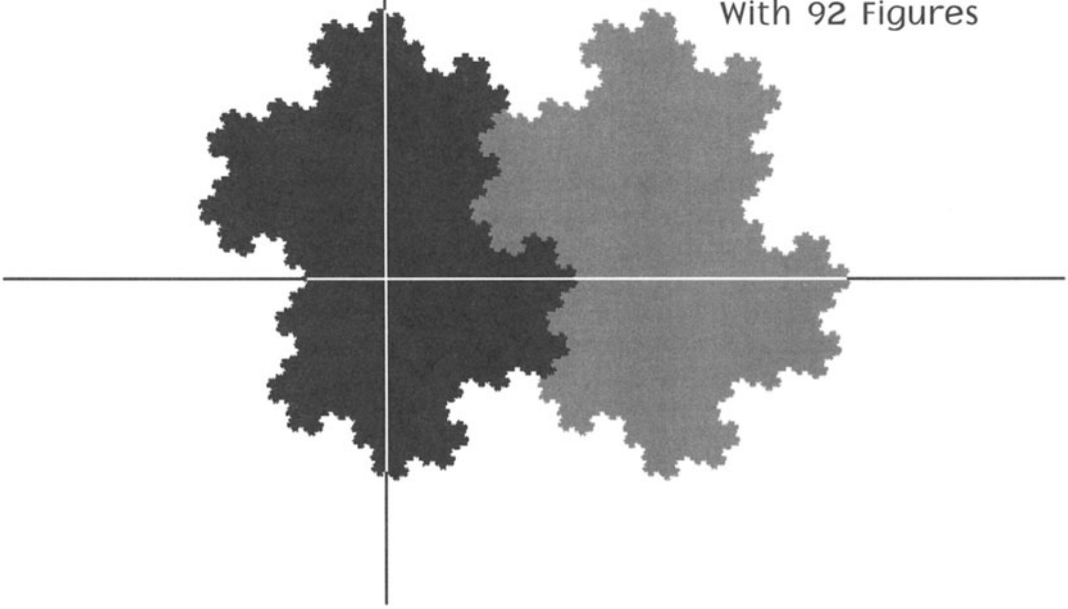
The Scalable Structure of Information

With 92 Figures



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For Joan and Rena,

who fortunately knew not to hold their breath.

Preface

The authors have been beguiled and entranced by mathematics all of their lives, and both believe it is the highest expression of pure thought and an essential component—one might say the *quintessence*—of nature. How else can one explain the remarkable effectiveness of mathematics in describing and predicting the physical world?

The projection of the mathematical method onto the subspace of human endeavors has long been a source of societal progress and commercial technology.[1] The invention of the electronic digital computer (not the mechanical digital computer of Babbage) has made the role of mathematics in civilization even more central by making mathematics active in the operation of products.

The writing of this book was intertwined with the development of a start-up company, Aware, Inc. Aware was founded in 1987 by one of the authors (H.L.R.), and the second author (R.O.W.) put his shoulder to the wheel as a consultant soon after.

The founding vision of Aware was a company that would develop technology-based products that depended in some essential way on advanced mathematics, that is, a *mathematical engineering company*. From the beginning, we thought of mathematical engineering as a discipline and mathematical engineering companies as primary developers of new technology that would come to characterize the

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[1] See our previous book *Mathematics in Civilization* [150] for an overview of the long-standing and often amazing interaction of civilization and mathematics.

twenty-first century as products become increasingly dependent on active mathematical algorithms and too complex to be physically testable.[2]

It seemed clear that the company would be required to create new mathematics as well as to apply known methods and results. This increased risks for investors and stretched out product-development schedules. It called for a team of employees whose abilities could have entitled them to university careers as research mathematicians, computer scientists, and, eventually, communication engineers in reputable universities. It had other consequences that were less obvious at the time: as ideas evolved from concept to marketable products and the company changed from a start-up to an established public company subject to the rule of Wall Street and its customers, the role of mathematicians and other theoreticians in the company diminished, and their status became anomalous.

H.L.R.'s original plan was based on previous experience at Thinking Machines Corporation, where he learned about the relative effectiveness of silicon and algorithms in performing computations. Although the performance of integrated circuits doubled every 18 months,[3] it seemed that perhaps half of the performance gains in mathematical computation could be attributed to the invention of mathematical algorithms that were well adapted to digital computation.[4] A third, less systematic, factor was the computational efficiency of the *mathematical representation* used to represent the underlying physical process. If a high-performance digital implementation, a computationally efficient algorithm, and an efficient mathematical representation could be combined, one would have the most efficient way to solve the problem.

The mathematical representation depends on the nature of the product functionality to an extent that could make one wonder whether there is enough commonality to enable one representational strategy to suffice for many different problems. Here, the effectiveness of *positional notation* for numbers provides an ancient but central example. It was easy to believe that a generalization of positional notation from the representation of numbers to the representation of functions had the power and generality to cause what today is called a "paradigm shift" in the way mathematics is used to solve practical problems.[5] This rather abstract notion was reinforced when H.L.R. heard Edward Adelson and Peter Burt, then at the Sarnoff Laboratories, describe their work on the *pyramid* representations of images. For images, a pyramid representation captures the essence of a kind of "positional notation" for image data cloaked in the garb of "multiresolution analysis." This important example was enough to convince H.L.R. that there were fundamental new mathematical representations that could represent the data for problems of practical importance very much more efficiently than conventional representations such as the Fourier or power series. These ideas led to his development of the

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[2] Chapter 1 discusses some of these extramathematical questions.

[3] Which it continues to do to this day.

[4] The Fast Fourier Transform is perhaps the best-known example.

[5] The multiresolution nature of positional notation is discussed in Chapter 3.

higher-dimensional bases generalizing the work of Haar from 1910 to lattices associated with quadratic number fields,[6] and to the search for capital to start Aware. H.L.R. was also convinced that a mathematical engineering company based on this approach to the representation problem and devoted to image processing and the numerical solution of partial differential equations made business sense.

After Aware was formed, these ideas and their applications were implemented in more than 100,000 lines of operational software. Then Daubechies's preprint [39] appeared. This seminal paper created the general theory of compactly supported wavelets of one variable. It was immediately clear that this paper achieved the goal of a "positional representation" for functions in a general and a profound way; it was just what was needed to achieve the company's objectives. It took very little time to adapt our existing software to this new mathematical environment. The company's practical orientation encouraged us to extend the theory in various directions motivated by the possible applications we envisioned.

In the earliest days, it was completely clear to people at Aware what the long-term practical implications of wavelets would be. We had been thinking about the applications of "a positional notation for functions" for some time, and understood from the start that this was one of the most important mathematical developments of the last quarter of the twentieth century. Soon after the initial implementation of these ideas at Aware and as a significant literature devoted to these ideas by researchers around the world evolved, this book was planned in the midst of an active research and commercial development program. Many years later, we have finally brought together in the current book ideas devoted to the development of the theory of compactly supported wavelets and multiresolution analysis, as well as some of its applications to image processing, numerical analysis, and telecommunications.

In 1990, DARPA (Defense Advanced Research Projects Agency), a U.S. Department of Defense agency noted for its forward-looking support of technology, established a five-year program of support for wavelet research. The initial goals were to stimulate the development, primarily at research universities, of a technical infrastructure in wavelet mathematics, and to identify potential applications where wavelet methods might yield "breakthrough" performance improvements. Aware already had an extensive cadre of trained personnel focused on wavelet analysis; it received one of the first and largest contracts. This and other government research and development contracts gave Aware the opportunity to explore a much broader field of potential markets than would have been possible based on private investment alone; in effect, these research contracts extended the company's investment capital.

The results of these investments are reflected throughout this book, at the abstract level as well as in the applications. By the early 1990s, it had become clear that wavelets could provide a heretofore unattainable level of performance in image compression and in digital modulation for telecommunications. At that point, the

[6]The general theory is discussed in Chapter 7.

company decided to concentrate its efforts in telecommunications and to maintain a secondary commercial effort in high-quality image compression.

In the applications arena, Aware's innovations include the first multiresolution wavelet integrated circuit; a new quality level of still image compression;[7] the first commercial wavelet-based nonlinear video editing workstation, driven by Aware's wavelet chipset and compression software; the first high-quality commercial wavelet video compression algorithms; a wavelet-based special-purpose processor board for multiresolution solution of the two-dimensional heat equation; a general wavelet method for the numerical solution of nonlinear partial differential equations;[8] commercial wavelet audio compression that put more than seven hours of high-quality audio on one CD-ROM; commercial wavelet image compression for medical radiology and geophysical images; and commercial wavelet modulation for high-speed xDSL and hybrid-fiber coax modems.[9] These innovations and others helped motivate much of the work in this book.

One of authors' shared dreams was the establishment of a mathematical engineering laboratory at Rice University, the academic home of one of us (R.O.W.). This came to pass, and the Computational Mathematics Laboratory (CML) at Rice University was founded with an initial University Research Initiative grant from DARPA in 1991. Today, this research laboratory is involved in fundamental research in wavelet analysis and develops wavelet-based tools for applications in the areas of radar and sonar imaging, geophysics, and medical imaging. Many of the ideas in this book were developed at Aware and by mathematicians and electrical engineers at CML both jointly and individually.

As the authors of this book, we owe many debts to many people. Foremost are people associated with Aware and CML in one capacity or another.[10] This includes the mathematicians, computer scientists, engineers, investors, and others associated with Aware at various times over the past decade who made the company possible and who contributed to the development of this new intellectual framework of wavelet analysis as well as to the new technology that it spawned. Moreover, the university faculty, the postdoctoral fellows, and the graduate and undergraduate students associated with CML at Rice over the past eight years have immensely helped in the creation of this book and many of the ideas in it.

Specifically, the authors would like to thank the mathematicians Wayne Lawton, Peter Heller, David Pollen, John Weiss, Andrew Latto, Stephen Del Marco, and Myoung An; the computer scientists John Huffman, David Linden-Plummer, and Aaron Cohen; the communication engineers Michael Tzannes, Stuart Sandberg, Marcos Tzannes, Richard Gross, Edward Reiter, John Stautner, and Halil Padir, all members of the staff at Aware. We want to thank Aware's academic consultants: Sidney Burrus (Rice University), Roland Glowinski (University of Houston),

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[7]Discussed in Chapter 13.

[8]Discussed in Chapters 10–12.

[9]Discussed in Chapter 14.

[10]The attentive reader will notice that we seem to owe debts to ourselves; actually, each author owes a large debt to the other.

Michael Godfrey (Imperial College, London and Stanford), Chao Lu (Baltimore), John Proakis (Northeastern), Gilbert Strang (MIT), Richard Tolimieri (CUNY), and Truong Nguyen (Wisconsin), whose intellectual collaboration and support made a big difference in this enterprise, especially in the early days, when it was less clear where this vision was going to go. H.L.R. would like to express special thanks to Wayne Lawton for his collaboration in Chapter 7 and to Michael Tzannes for his contributions to Chapter 14, and to both for many productive discussions. R.O.W. would like to acknowledge his debt to Peter Heller from whom he has learned so much in their extensive collaboration and who contributed extensively to Chapters 4, 5, and 6.

Richard Baraniuk, Sidney Burrus, and William Symes at Rice and Roland Glowinski at the University of Houston have been faculty members of CML, and their efforts on its behalf are much appreciated. R.O.W. would like to thank Richard Baraniuk, in particular, for his thoughtful reading of an early draft of this book. We would like to thank Xiao-dong Zhou, Andreas Rieder, and T.W. Pan, who were postdoctoral fellows at CML at various times and who greatly contributed to this book. Hubert Bray, Clark Bray, and Kent McCormick were undergraduate students in CML who learned quickly and contributed a great deal. Yuan Wang and Haitao Guo were graduate students at CML who contributed to the book in a number of different ways. We want to thank especially Ramesh Gopinath, who was a graduate student at Rice, a visiting researcher while a student at Aware, and then a postdoctoral fellow at CML. R.O.W. would like to thank his collaborators Roland Glowinski, Xiao-dong Zhou, T.W. Pan, and Andreas Rieder, whose input contributed greatly to the chapter on partial differential equations. Jun Tian has played a very important role in this book, as a graduate student, as a postdoctoral fellow, and currently as a research scientist at Rice. R.O.W. would like to thank him personally for all the hard work he has done on this book.

During the year 1995–1996, R.O.W. was a visiting professor at the University of Bremen in Germany, and he would like to thank Heinz-Otto Peitgen for his very generous hospitality there at CeVis and MeVis, his two interrelated centers for visualization. We would like to thank Rolf Raber for some of the figures for the book, one of which is on its cover. Heinz-Otto Peitgen and Guentcho Skordev (of the Center for Complex Systems and Visualization—CeVis) used a rough draft of this book as a resource for a lecture course on wavelets, and we appreciate their feedback. We also want to thank Markus Lang, who was a postdoctoral fellow at CML and then a staff member at the newly founded Center for Medical Diagnostic Systems and Visualization (MeVis).

It is rare that investors have the patience and the means to fund a start-up that must build its technology before it can define its market and products.[11] Aware was fortunate in its investors, who had their own vision and an exceptional degree of

[11]This strategy is not recommended. It is not that Aware's investors didn't know better; they had a greater vision and a commensurate tolerance for risk. Genetic engineering start-ups may offer one of the few valid comparisons.

patience. They played a critical role, for they provided the lifeline while the company developed products based on the new mathematical technology. Three investors deserve special mention and appreciation: Charles K. Stewart, John Stafford, and John Kerr. Today, all three are members of the board of directors of Aware, and Mr. Stewart is chairman.

Almost as important as patient investors are patient editors. We owe a great debt to the editorial staff of Springer-Verlag, and, in particular, to Ulrike Schmickler-Hirzebruch and Thomas von Foerster. We are particularly grateful to Rena Wells of Rosenlaui Research in Houston, Texas, who has contributed immensely to the book with her editing and typesetting skills, and whose patience over the many years of work on this book is admirable.

Finally, a thought for the reader: Wavelet analysis is an emerging mathematical discipline. We hope that the structure presented here will provide the reader with a steady framework that will help to organize the swirl of future developments.

Boston, Massachusetts
Houston, Texas

Howard L. Resnikoff
Raymond O. Wells, Jr.

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