Applied Mathematical Sciences

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Applied Mathematical Sciences | Volume 57

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(continued on inside back cover)

H.-J. Reinhardt

Analysis of Approximation Methods for Differential and Integral Equations

With 20 Figures

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H.-J. Reinhardt
Johann-Wolfgang-Goethe-Universität
Fachbereich Mathematik
6000 Frankfurt Main
Federal Republic of Germany

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Preface

This book is primarily based on the research done by the Numerical Analysis Group at the Goethe-Universität in Frankfurt/Main, and on material presented in several graduate courses by the author between 1977 and 1981. It is hoped that the text will be useful for graduate students and for scientists interested in studying a fundamental theoretical analysis of numerical methods along with its application to the most diverse classes of differential and integral equations.

The text treats numerous methods for approximating solutions of three classes of problems: (elliptic) boundary-value problems, (hyperbolic and parabolic) initial value problems in partial differential equations, and integral equations of the second kind. The aim is to develop a unifying convergence theory, and thereby prove the convergence of, as well as provide error estimates for, the approximations generated by specific numerical methods. The schemes for numerically solving boundary-value problems are additionally divided into the two categories of finite-difference methods and of projection methods for approximating their variational formulations.

In accordance with our aims, we present in Part I approximation methods to each of the aforementioned classes of problems, state results concerning the solvability of the underlying approximate equations, and, for nonlinear problems, consider iterative procedures for their solution. Then, in Part II, we develop our underlying convergence theory for sequences of equations based on the concept of "discrete convergence". In Part III and IV, we reconsider the problem areas mentioned above and show, by means of our theory, the convergence of solutions obtained by specific methods when applied to a series of examples.

The convergence theory of approximation methods that we present in the text is applicable to a series of classes of both linear and nonlinear problems and will in many cases enable us to obtain two-sided error estimates. The methods we consider, for example, encompass finite element methods as well as finite-difference approximations for both ordinary and partial differential equations. Similarly, projection methods and methods based on quadrature formulas for numerically treating integral equations of the second kind can be analyzed with the techniques presented here. Moreover, the general convergence results can still be applied to other problems and other classes of approximation methods (to, say, initial value problems in ordinary differential equations, collocation methods for initial and boundary-value problems, etc.). The general convergence theory presented in the text was essentially developed by F. Stummel. Further developments and refinements were made by R. D. Grigorieff and his group (at the Technical University in Berlin) and by the author. At some places in the text, in particular in Part IV, unpublished results are contained in the presentation. It is appropriate, at this point, to mention the earlier contributions of Aubin (1967), Browder (1967), Céa (1964), Pereyra (1967), Petryshyn (1967b, 1968a), Stetter (1965a, 1965b, 1966), Vainikko (1967) in developing a convergence theory for approximation methods.

By necessity, we must limit the scope of the material presented in this book. The convergence theory, on the one hand, is developed in a very general setting, but, on the other, is restricted to problems where the approximating equations are expressed in terms of equicontinuously equidifferentiable mappings. These, of course, include linear problems. In the concrete applications, we mostly study problems with one spatial dimension. In higher dimensional problems, however (e.g. Poisson's equation or the two-dimensional heat equation), we consider only examples having rectangular spatial domains. We would like to mention that the approximation theory of finite elements can be treated by the analysis we develop in this book; but, due to the basic orientation of our presentation, we shall study finite element methods only in the context of specific examples. Moreover, there are numerous variants of the schemes considered in the text which will not be discussed because of lack of space. The concrete methods we consider serve to demonstrate the applicability of our general convergence theory as well as provide analytical techniques.

This book may also serve as a reference for a series of well-known and other numerical schemes for the problem classes considered. For practical purposes, the numerical methods can be chosen - and used - according to their stability and convergence properties provided in the text. It should be noted, however, that one and the same method may be stable, inversely stable and convergent or may not be stable, etc. depending on the norms underlying the analysis. For example, inverse stability of the well-known Crank-Nicolson-Galerkin method approximating the heat equation is explored three-fold in the text with the result that this method is conditionally stable with respect to the maximum norm, unconditionally stable in the sense of the von Neumann stability criterion, and unconditionally stable relative to suitable Sobolev norms which are even stronger than the maximum norms. Furthermore, there are schemes which produce converging approximations but only for restricted classes of problems; this phenomenon is expressed by the concept of stable convergence. We like to emphasize again that any stability or convergence statement is only relative to the underlying norms. The interested, more practically orientated reader is invited to study such phenomena by means of computational experiments which, for most of the examples considered in the book, can be performed on personal computers.

We now want to give some technical hints which should be noted in order to make the reading of the text easier. The book consists of thirteen chapters and is organized in four parts. Each chapter contains different sections and is preceeded by an introduction. The same notation will be used for labeling formulas and specifying conditions; we refer to, e.g., formula (60) in Chapter 4 simply by 4.(60). A different notation will be employed for denoting theorems, lemmas, important properties, and propositions, e.g., Theorem 5.9, Lemma 6.3, Property 7.6, Proposition 12.8, etc.. In the text and at the conclusion of each chapter, we cite references only by author and year of publication, and give additional works pertinent to the study but not specifically referred to in the text. The full reference complete with title of the cited work can be found in the bibliography following the final chapter. At a few places in the text, comments are made concerning extensions of the results we present but, in general, we do not give an extensive discussion of related literature. The reader who is not interested in all problem areas in the text should select the relevant chapters according to the following diagram:

Problems	Chap. in Part I	Chap.	
Boundary-value problems	1	8	Part III
Variational equations	2	9	Part III
Integral equations	3	10	Part III
Initial value problems	4	11-13	Part IV

In order to appreciate the convergence analysis in Chapters 8 to 13, it is, however, necessary to study - or, at least, to take notice of - the convergence theory developed in Part II (Chapters 5-7). We strongly recommend though that the reader has a basic knowledge of numerical analysis and functional analysis in order to gain the most benefit from this book.

The author would like to acknowledge his deep indebtedness to Professor F. Stummel who stimulated his interest in and from whom he learned about numerical analysis, first as a student and later as a collaborator. For various improvements, such as shorter proofs and better exposition at various places, the author is especially obliged to Professor R. D. Grigorieff who has been kind enough to read most of the manuscript. The author would like to express his appreciation to Professor

H. D. Victory, Jr., for his careful translation of - and, in some cases, suggestions for improving - the original German manuscript. For reading and discussing several chapters, the author is indebted to Professor K. H. Müller, Professor I. Sloan, and Privatdozent J. Lorenz, and to some of his students for reading parts of earlier versions of the manuscript. Special thanks are due to Mrs. H. Meßner for her prompt preparation of a preliminary version of this book, and to Mrs. Kate MacDougall for her careful typing of the final copy.

Contents

Page