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continued after index

Jeffrey Rauch

Partial Differential Equations

With 42 Illustrations

Springer

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Preface

This book is based on a course I have given five times at the University of Michigan, beginning in 1973. The aim is to present an introduction to a sampling of ideas, phenomena, and methods from the subject of partial differential equations that can be presented in one semester and requires no previous knowledge of differential equations. The problems, with hints and discussion, form an important and integral part of the course.

In our department, students with a variety of specialties—notably differential geometry, numerical analysis, mathematical physics, complex analysis, physics, and partial differential equations—have a need for such a course.

The goal of a one-term course forces the omission of many topics. Everyone, including me, can find fault with the selections that I have made.

One of the things that makes partial differential equations difficult to learn is that it uses a wide variety of tools. In a short course, there is no time for the leisurely development of background material. Consequently, I suppose that the reader is trained in advanced calculus, real analysis, the rudiments of complex analysis, and the language of functional analysis. Such a background is not unusual for the students mentioned above. Students missing one of the "essentials" can usually catch up simultaneously.

A more difficult problem is what to do about the Theory of Distributions. The compromise which I have found workable is the following. The first chapter of the book, which takes about nine fifty-minute hours, does not use distributions. The second chapter is devoted to a study of the Fourier transform of tempered distributions. Knowledge of the basics about $\mathscr{D}(\Omega)$, $\mathscr{E}(\Omega)$, $\mathscr{D}'(\Omega)$, and $\mathscr{E}'(\Omega)$ is assumed at that time. My experience teaching the course indicates that students can pick up the required facility. I have provided, in an appendix, a short crash course on Distribution Theory. From Chapter 2 on, Distribution Theory is the basic language of the text, providing a good setting for reinforcing the fundamentals. My experience in teaching this course is that students have less difficulty with the distribution theory than with geometric ideas from advanced calculus (e.g. $d\varphi$ is a one-form which annihilates the tangent space to $\{\varphi = 0\}$).

There is a good deal more material here than can be taught in one semester. This provides material for a more leisurely two-semester course and allows the reader to browse in directions which interest him/her. The essential core is the following:

Chapter 1. Almost all. A selection of examples must be made.

Chapter 2. All but the L^p theory for $p \neq 2$. Some can be left for students to read.

Chapter 3. The first seven sections. One of the ill-posed problems should be presented.

Chapter 4. Sections 1, 2, 5, 6, and 7 plus a representative sampling from Sections 3 and 4.

Chapter 5. Sections 1, 2, 3, 10, and 11 plus at least the statements of the standard Elliptic Regularity Theorems.

These topics take less than one semester.

An introductory course should touch on equations of the classical types, elliptic, hyperbolic, parabolic, and also present some other equations. The energy method, maximum principle, and Fourier transform should be used. The classical fundamental solutions should appear. These conditions are met by the choices above.

I think that one learns more from pursuing examples to a certain depth, rather than giving a quick gloss over an enormous range of topics. For this reason, many of the equations discussed in the book are treated several times. At each encounter, new methods or points of view deepen the appreciation of these fundamental examples.

I have made a conscious effort to emphasize qualitative information about solutions, so that students can learn the features that distinguish various differential equations. Also the origins in applications are discussed in conjunction with these properties. The interpretation of the properties of solutions in physical and geometric terms generates many interesting ideas and questions.

It is my impression that one learns more from trying the problems than from any other part of the course. Thus I plead with readers to attempt the problems.

Let me point out some omissions. In Chapter 1, the Cauchy–Kowaleskaya Theorem is discussed, stated, and much applied, but the proof is only indicated. Complete proofs can be found in many places, and it is my opinion that the techniques of proof are not as central as other things which can be presented in the time gained. The classical integration methods of Hamilton and Jacobi for nonlinear real scalar first-order equations are omitted entirely. My opinion is that when needed these should be presented along with symplectic geometry. There is a preponderance of linear equations, at the expense of nonlinear equations. One of the main points for nonlinear equations is their differences with the linear. Clearly there is an order in which these things should be learned. If one includes the problems, a reasonable dose of nonlinear examples and phenomena are presented. With the exception of the elliptic theory, there is a strong preponderance of equations with constant coefficients, and especially Fourier transform techniques. The reason for this choice is that one can find detailed and interesting information without technical complexity. In this way one learns the ideas of the theory of partial differential equations at minimal cost. In the process, many methods are introduced which work for variable coefficients and this is pointed out at the appropriate places.

Compared to other texts with similar level and scope (those of Folland, Garabedian, John, and Treves are my favorites), the reader will find that the present treatment is more heavily weighted toward initial value problems. This, I confess, corresponds to my own preference. Many time-independent problems have their origin as steady states of such time-dependent problems and it is as such that they are presented here.

A word about the references. Most are to textbooks, and I have systematically referred to the most recent editions and to English translations. As a result the dates do not give a good idea of the original publication dates. For results proved in the last 40 years, I have leaned toward citing the original papers to give the correct chronology. Classical results are usually credited without reference.

I welcome comments, critiques, suggestions, corrections, etc. from users of this book, so that later editions may benefit from experience with the first.

So many people have contributed in so many different way to my appreciation of partial differential equations that it is impossible to list and thank them all individually. However, specific influences on the structure of this book have been P.D. Lax and P. Garabedian from whom I took courses at the level of this book; Joel Smoller who teaches the same course in a different but related way; and Howard Shaw whose class notes saved me when my own lecture notes disappeared inside a moving van. The integration of problems into the flow of the text was much influenced by the *Differential Topology* text of Guillemin and Pollack. I have also benefited from having had exceptional students take this course and offer their criticism. In particular, I would like to thank Z. Xin whose solutions, corrections, and suggestions have greatly improved the problems. Chapters of a preliminary version of this text were read and criticized by M. Beals, J.L. Joly, M. Reed, J. Smoller, M. Taylor, and M. Weinstein. Their advice has been very helpful. My colleagues and coworkers in partial differential equations have taught me much and in many ways. I offer a hearty thank you to them all.

The love, support, and tolerance of my family were essential for the writing of this book. The importance of these things to me extends far beyond professional productivity, and I offer my profound appreciation.

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