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Functional Data Analysis

Second Edition

With 151 Illustrations

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Preface to the Second Edition

This book continues in the footsteps of the First Edition in being a snapshot of a highly social, and therefore decidedly unpredictable, process. The combined personal view of functional data analysis that it presents has emerged over a number of years of research and contact, and has been greatly nourished by delightful collaborations with many friends. We hope that readers will enjoy the book as much as we have enjoyed writing it, whether they are our colleagues as researchers or applied data analysts reading the book as a research monograph, or students using it as a course text.

As in the First Edition, live data are used throughout for both motivation and illustration, showing how functional approaches allow us to see new things, especially by exploiting the smoothness of the processes generating the data. The data sets exemplify the wide scope of functional data analysis; they are drawn from growth analysis, meteorology, biomechanics, equine science, economics and medicine.

“Back to the data” was the heading to the last section of the First Edition. We did not know then how well those words would predict the next eight years. Since then we have seen functional data applications in more scientific and industrial settings than we could have imagined, and so we wanted an opportunity to make this new field accessible to a wider readership than the the first volume seemed to permit. Our book of case studies, Ramsay and Silverman (2002), was our first response, but we have known for some time that a new edition of our original volume was also required.

We have added a considerable amount of new material, and considered carefully how the original material should be presented. One main objec-

tive has been, especially when introducing the various concepts, to provide more intuitive discussion and to postpone needless mathematical terminology where possible. In addition we wanted to offer more practical advice on the processing of functional data. To this end, we have added a more extended account of spline basis functions, provided new material on data smoothing, and extended the range of ways in which data can be used to estimate functions. In response to many requests, we have added some proposals for estimating confidence regions, highlighting local features, and even testing hypotheses. Nevertheless, the emphasis in the revision remains more exploratory and confirmatory.

Our treatment of the functional linear model in the First Edition was only preliminary, and since then a great deal of work has been done on this topic by many investigators. A complete overhaul of this material was called for, and the chapters on linear modelling have been completely reworked. On the other hand, our coverage of principal components analysis and canonical correlation still seems appropriate, and not much has been changed. Readers reacted to the later chapters on differential equations as being difficult, and so we have tried to make them a friendlier place to be.

In some places we have opted for an ‘intuitive’ rather than ‘rigorous’ approach. This is not merely because we want our book to be widely accessible; in our view the theoretical underpinnings of functional data analysis still require rather more study before a treatment can be written that will please theoreticians. We hope that the next decade will see some exciting progress in this regard.

We both believe that a good monograph is a personal view rather than a dry encyclopedia. The average of two personal views is inevitably going to be less ‘personal’ than either of the two individual views, just as the average of a set of functions may omit detail present in the original functions. To counteract this tendency, we have ensured that everything we say in our informal and intuitive discussion of certain issues is the view of at least one of us, but we have not always pressed for unanimous agreement!

We owe so much to those who helped us to go here. We would like to repeat our thanks to those who helped with the First Edition: Michal Abrahamowicz, Philippe Besse, Darrell Bock, Catherine Dalzell, Shelly Feran, Randy Flanagan, Rowena Fowler, Theo Gasser, Mary Gauthier, Vince Gracco, Nancy Heckman, Anouk Hoedeman, Steve Hunka, Iain Johnstone, Alois Kneip, Wojtek Krzanowski, Xiaochun Li, Kevin Munhall, Guy Nason, Richard Olshen, David Ostry, Tim Ramsay, John Rice and Xiaohui Wang. We also continue our grateful acknowledgement of financial support from the Natural Science and Engineering Research Council of Canada, the National Science Foundation and the National Institute of Health of the USA, and the British Engineering and Physical Sciences Research Council. The seed for the First Edition, and therefore for the Revised Edition as well, was planted at a discussion meeting of the Royal Statistical Society

Research Section, where one of us read a paper and the other proposed the vote of thanks, not always an occasion that leads to a meeting of minds!

Turning to the Second Edition, Sofia Mosesova and Yoshio Takane read the entire manuscript with an eye to the technical correctness as well as the readability of what they saw, and caught us on many points. David Campbell helped with the literature review that supported our “Further readings and notes” sections. Time spent at the University of British Columbia made possible many stimulating conversations with Nancy Heckman and her colleagues. A discussion of many issues with Alois Kneip as well his hospitality for the first author at the University of Mainz was invaluable. The opportunity for us to spend time together afforded by St Peter’s College and the Department of Statistics at Oxford University was essential to the project.

April 2005

Jim Ramsay & Bernard Silverman

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