

Murray H. Protter

# Basic Elements of Real Analysis

With 48 Illustrations



Springer

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*Front cover illustration:*  $f_n$  converges to  $f$ , but  $f_0^1 f_n$  does not converge to  $f_0^1 f$ . (See p. 167 of text for explanation.)

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