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Stochastic Calculus

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Contents

Preface

In the axioms of probability theory proposed by Kolmogorov the basic "probabilistic" object is the concept of a probability model or probability space. This is a triple $(\Omega,\ \mathcal{F},\ \mathbf{P})$, where Ω is the space of elementary events or outcomes, $\mathcal{F}$ is a σ-algebra of subsets of Ω announced by the events and $\mathbf{P}$ is a probability measure or a probability on the measure space $(\Omega,\ \mathcal{F})$. This generally accepted system of axioms of probability theory proved to be so successful that, apart from its simplicity, it enabled one to embrace the classical branches of probability theory and, at the same time, it paved the way for the development of new chapters in it, in particular, the theory of random (or stochastic) processes.

In the theory of random processes, various classes of processes have been studied in depth. Theories of processes with independent increments, Markov processes, stationary processes, among others, have been constructed. In the formation and development of the theory of random processes, a significant event was the realization that the construction of a "general theory of random processes" requires the introduction of a flow of σ-algebras (a filtration) $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$ supplementing the triple $(\Omega,\ \mathcal{F},\ \mathbf{P})$, where $\mathcal{F}_t$ is interpreted as the collection of events from $\mathcal{F}$ observable up to time t.

It is this assumption of the presence on $(\Omega, \mathcal{F}, \mathbf{P})$ of a flow $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$ that has given rise to such objects as Markov times or stopping times, adapted processes, optional and predictable σ-algebras, martingales, local martingales, semimartingales, the stochastic integral, the Itô change of variables formula, etc., which are the ingredients of the theory of stochastic calculus.

Thus, stochastic calculus axiomatizes the concept of a stochastic basis

$$\mathcal{B} = (\Omega,\ \mathcal{F},\ \mathbf{F} = (\mathcal{F}_t)_{t\geq 0},\ \mathbf{P}),$$

which lies at the basis of our entire discussion. Here $(\Omega,\ \mathcal{F},\ \mathbf{P})$ is a probability space, and $\mathbf{F}$ is some distinguished flow of σ-algebras.

The formation of the concept of a "stochastic basis" as one of the specifications of a probability space passed through many stages of discussion of particular cases, refinements, generalizations, etc. Here the theory of stochastic integration for Brownian motion and a centered Poisson measure, developed by Itô, was crucial.

In the first chapter of this volume, an introduction is given to stochastic calculus which is called upon for presenting various aspects of Brownian motion and its connection with the theory of partial differential equations, the latter being the fundamental principle of Kolmogorov's classical paper "Analytical Methods in Probability Theory".

Stochastic integration theory has been brought to perfection for the most part in random processes that are solutions of stochastic differential equations which, as we have said, are a particular case of semimartingales — that wide class of random processes for which stochastic calculus gives a powerful method of analysis.

The second chapter is devoted specifically to the theory of stochastic differential equations as well as stochastic evolution equations and the stochastic calculus of variations (or the Malliavin calculus), a highly effective probabilistic apparatus for study in the theory of partial differential equations, theoretical physics, and ergodic theory.

The third chapter is devoted to the general theory of stochastic calculus proper on probability spaces with filtrations. It presents the basic elements of the general theory of random processes, stochastic integration over semimartingales, and a number of their applications.

The ideas and methods of stochastic calculus traditionally have found and still find application in diverse sections of probability theory and mathematical statistics. This is illustrated, in particular, in the fourth chapter, in which the methods of martingale theory and stochastic calculus are applied to study questions of the weak convergence of random processes considered as random elements with values in metric spaces.

The contributions made by the team of authors of this volume, S.V. Anulova, A.Yu. Veretennikov, N.V. Krylov, R.Sh. Liptser and A.N. Shiryaev are as follows: Chap. 1 was written by N.V. Krylov; Chap. 2, Part I, §§1, 3, 4; Chap. 2, Part II; Chap. 2, Part III were written by A.Yu. Veretennikov; Chap. 2, Part I, §§2, 5, 6 were written by S.V. Anulova; Chap. 3 was written by R.Sh. Liptser and A.N. Shiryaev; Chap. 4, Part II was written by R.Sh. Liptser; Chap. 4, Part I was written by A.N. Shiryaev.

A.N. Shiryaev