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Positive Polynomials

From Hilbert's 17th Problem
to Real Algebra

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Preface

Exactly 100 years ago, at the turn of the 19th to the 20th century, in his famous address to the 1900 International Congress of Mathematicians, David Hilbert [1900] presented a list of 23 problems that he considered to be the most important problems left from the old century to be solved in the new one. The 17th problem, in its simplest form, is as follows:

> *Suppose $f \in \mathbb{R}[X_1, \ldots, X_n]$ is a real polynomial in n indeterminates, and $f(x) \geq 0$ for all $x \in \mathbb{R}^{(n)}$. Does there then necessarily exist a representation of f as a sum of squares of real rational functions, i.e., in the form*
>
> $$f = \sum_i r_i^2,$$
>
> *for finitely many r_i from the field $\mathbb{R}(X_1, \ldots, X_n)$ of rational functions in $X_1, \ldots, X_n$?*

It did not take long for the problem to be solved: in [1926] E. Artin presented a quite remarkable solution to the problem. Rather than constructing a representation of f as a sum of squares of rational functions, Artin showed the mere existence of such a representation, by an indirect proof. Nevertheless, the solution offered a "global" characterization of positivity of polynomials on $\mathbb{R}^{(n)}$.

This brings us to the main goal of our book: we seek characterizations of those polynomials f that are positive on certain sets, themselves defined by polynomial inequalities. In every case, these characterizations consist of representing f within the ring of all real polynomials in such a way that the required positivity of f is reflected instantly. Many results of this type have been obtained over the last 75 years, all starting with Artin's solution of Hilbert's 17th problem. New methods have been developed over the years, focusing on "reality" and "positivity." In a sense, Artin's solution may be understood as the beginning of "real algebra." Thus, not surprisingly, the second goal of this book is to present an introduction to real algebra.

The book is based on a two-semester course having exactly these two goals; it was given by the first author at the University of Konstanz during the summer semester of 1999 and the winter semester of 1999–2000. The present form of the book arose during a joint stay by both authors at the

Mathematical Research Institute in Oberwolfach (Germany), under its "Research in Pairs" program.

The part of the book that constitutes an introduction to real algebra consists of:

Chapter 1, where we introduce the theory of ordered fields and real closures of such fields (1.1–1.3);

Chapter 2, where we give an introduction to semialgebraic sets and Tarski's Transfer Principle (2.1–2.4);

Chapter 3, where we present a short introduction to the theory of the Witt ring of a field K, and study the total signature map on the space of orderings of K—the "real spectrum" of K (3.1–3.3);

Chapter 4, where we introduce the real spectrum of an arbitrary commutative ring, and give a special description of the real spectrum of the particular ring $\mathbb{R}[X_1, \dots, X_n]$ of real polynomials (4.1, 4.2, 4.4, 4.5); and

Chapter 5, where we study rings in which every element is bounded on the real spectrum, and give representations (i.e., homomorphisms) of such rings into rings of continuous real-valued functions on some compact Hausdorff space (5.1–5.4).

Our main goal—the improvements in the representation of f—is explained in the Introduction, and pursued in Chapters 5 to 8. Artin's solution of Hilbert's 17th problem is presented in Section 2.1 (Theorem 2.1.12). Generalizations of this problem, as well as improvements in the representation, are found in Sections 3.5, 4.2, 5.2, 5.3, 5.4, 6.3, 7.3, 8.3, and 8.4.

Each chapter has a section of exercises that may help the reader better understand what was treated in that chapter, and obtain some further information. Finally, each chapter ends with "bibliographical and historical comments," in which we try to inform the reader about the origins of the notions and results in that chapter, and their connections to other work.

We are most grateful to Markus Schweighofer, a Ph.D. student at the University of Konstanz, who contributed many of the exercises in the book. He also carefully read all drafts of the book, offering many corrections, clarifications, and improvements.

Konstanz, Germany, October 2000 *Alexander Prestel*
Baton Rouge, U.S.A., October 2000 *Charles N. Delzell*

Please visit the book's web site, containing errata, updates, and other material: `http://www.math.lsu.edu/~delzell/positive_updates.html`. And please send any corrections or suggestions that you may have to `alex.prestel@uni-konstanz.de` or `delzell@math.lsu.edu`.

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