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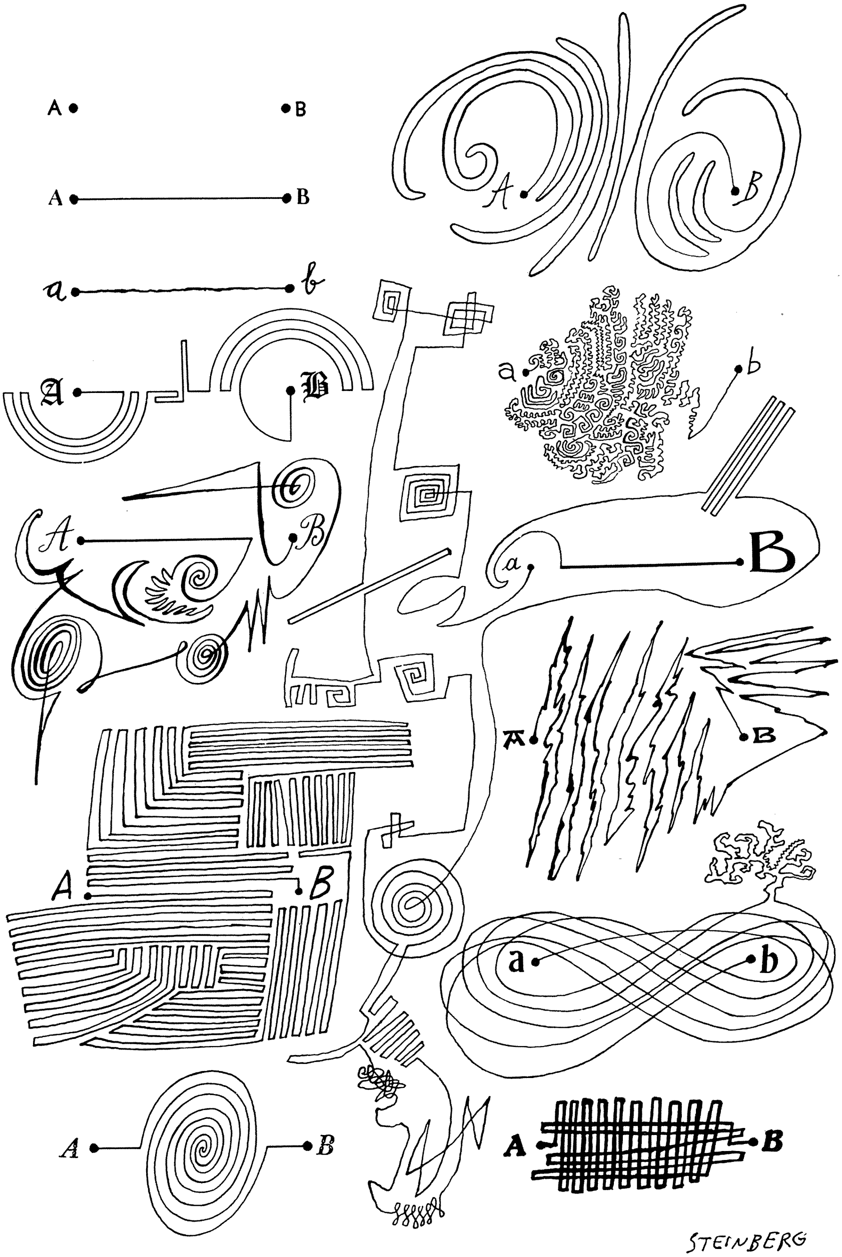

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Join Geometries

A Theory of Convex Sets and Linear Geometry

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To Sophie

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Introduction

The main object of this book is to reorient and revitalize classical geometry in a way that will bring it closer to the mainstream of contemporary mathematics. The postulational basis of the subject will be radically revised in order to construct a broad-scale and conceptually unified treatment.

The familiar figures of classical geometry—points, segments, lines, planes, triangles, circles, and so on—stem from problems in the physical world and seem to be conceptually unrelated. However, a natural setting for their study is provided by the concept of *convex set*, which is comparatively new in the history of geometrical ideas. The familiar figures can then appear as convex sets, boundaries of convex sets, or finite unions of convex sets. Moreover, two basic types of figure in linear geometry are special cases of convex set: *linear space* (point, line, and plane) and *halfspace* (ray, halfplane, and halfspace). Therefore we choose convex set to be the central type of figure in our treatment of geometry. How can the wealth of geometric knowledge be organized around this idea? By definition, a set is convex if it contains the segment joining each pair of its points; that is, if it is closed under the operation of joining two points to form a segment. But this is precisely the basic operation in Euclid. Our point of departure is to take the operation of joining two points to form a segment as fundamental, and to throw the burden of unifying the material on the consistent and relentless exploitation of this operation.

The postulates then will not involve complex ideas or complicated figures, but will state elementary properties of the join operation that can be grasped intuitively and verified concretely in planar diagrams.

The postulates are formulated as *universal* properties of points. Thus, there are no exceptional or degenerate cases to be excluded. This is in striking contrast with classical Euclidean postulates, such as: two *distinct* points determine a line; three *noncollinear* points determine a plane. As a result, proofs usually involve the application of the postulates as *general* principles and there is little or no need to consider the special or degenerate cases that arise so often in conventional treatments of Euclidean geometry.

A salient feature of the treatment is its freedom from the classical restriction to the study of geometries that are, at most, three-dimensional. Indeed, our postulates are dimension free—they involve no dimensionality assumption, explicit or implicit.

Consequently a major portion of the development is dimension free and is applicable to spaces of arbitrary dimension, finite or infinite. (This belies a widespread belief that the only effective way to study higher dimensional geometry is by the intervention of linear algebra.)

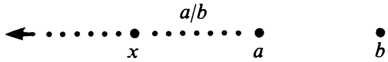
How does the theory compare with Euclidean geometry? The postulates are abstracted from Euclidean propositions, and the theory may be considered a generalization of Euclidean geometry. It is, however, much broader —the theory has been freed from constraints that arose naturally in the historical evolution of Euclidean geometry but now impede its development. Many familiar Euclidean propositions—in addition to the dimensional restriction mentioned above—are omitted from the postulate set. These include: (i) the Euclidean parallel postulate; (ii) the proposition that of three distinct collinear points, one is between the other two; and (iii) two distinct points determine a line. Moreover, the treatment is nonmetrical—no postulates for congruence have been assumed.

Can this brave new geometrical world be achieved merely by referring to a segment as the join of two points? Of course not. A reanalysis and reconstruction of classical geometry in terms of the join operation is required. First of all the join operation is not to be restricted artificially, it must apply equally well to all pairs of points, distinct *or* coincident. Even more important, the operation must be generalized to apply to all pairs of geometric figures.

In a Euclidean geometry, we define the *join* of points a, b, denoted by $a \cdot b$ or ab, to be the open segment with endpoints a and b if $a \neq b$ and define the join of a point a and itself to consist of a. The operation join is extended to apply to any two figures A and B in a natural way: The join $A \cdot B$ or AB, of A and B, is the union of all joins ab where point a ranges over A and point b over B.

There is, in Euclidean geometry, a second important operation—that of extending a segment indefinitely to form a ray. This can be treated as a sort of inverse operation to join and suggests the following

Definition. Let a and b be any points. Then a/b, the *extension* of *a from b*, is the set of points x which satisfy the condition that bx contain point a.



This operation is extended in the same way as join to define A/B, the *extension* of A *from* B, for any two figures A and B.

Chapter 1 provides an introduction to the abstract theory by studying the join and extension operations in Euclidean geometry in a concrete, intuitive, exploratory manner.

The formal development of the theory begins in Chapter 2, and is based on the idea of join operation. Let J be a set of elements (points) and $\cdot$ an operation which assigns to each ordered pair (a, b) of elements of J a uniquely determined subset of J, denoted by $a \cdot b$ or ab. Then the operation $\cdot$ is a *join operation* (in set J) and $a \cdot b$ is the *join* of a and b. We assume that the set J and the join operation $\cdot$ satisfy four postulates suggested by elementary properties of the Euclidean join operation. Convex sets are defined as closed under join, and their elementary properties deduced. The concepts of *geometrical* or *intrinsic* interior and closure of a convex set are defined. These ideas pervade the theory.

The convex hull of an arbitrary set is introduced in Chapter 3. Join theoretic formulas for convex hulls are derived. Polytopes are treated as convex hulls of finite sets.

In Chapter 4, the *extension* a/b of two elements a, b of J is defined, in the same way as Euclidean extension, in terms of join. Three new postulates involving extension are introduced to complete the basic postulate set for the theory. Speaking geometrically, the concept of *ray* (or *halfline*) is now available in the abstract theory. In formal terms, we have at our command an algebra, of strong deductive power, involving two "inverse" operations join $\cdot$ and extension $/$.

Chapter 5 introduces the idea of join geometry, our basic object of study. A *join geometry* is a model of the theory—it is a pair $(J, \cdot)$ composed of a set J and a join operation $\cdot$ in J, which satisfy the basic set of postulates. The notion of isomorphism of join geometries is studied. A collection of join geometries that are used as illustrative examples and counterexamples is presented. Real n-space $\mathbb{R}^n$ is converted into a join geometry by defining join in the natural manner. An infinite dimensional analogue of $\mathbb{R}^n$ is shown to contain a *pathological* convex set—a nonempty convex set whose interior is empty.

Chapter 6 studies *linear sets* (or *linear spaces*) defined as closed under join and extension. Among the topics considered are generation of linear sets, linear independence, and how *line* should be defined. It is interesting that linear sets of a join geometry bear analogies to subgroups of an abelian group.

Chapter 7 studies the idea of *extreme set of a convex set*. An extreme set of a convex set A is loosely a convex subset of A which is "peripheral" to A. The idea is suggested by, and is a generalization of, the classical notion of vertices, edges, and faces of a polyhedron. Two types of extreme sets, called components and faces, play important roles in the study of the structure of a convex set and are singled out for special study.

Chapter 8 deals with rays and halfspaces. A *ray* or *halfline* is defined as a set p/a, where p and a are points; p is its *endpoint*. Similarly, let L be a nonempty linear set, and a a point. Then L/a is a *halfspace* of L, or simply a *halfspace*; L is its *edge*. A study of rays is given, concentrating on rays with a common endpoint. This is generalized to an analogous treatment for halfspaces with a common edge. A halfspace of a linear set in a join geometry is analogous to a coset of a subgroup in an abelian group.

Chapter 9 presents a treatment of cones and hypercones based on the material of Chapter 8. A *cone* is the union of a family of rays that have a common endpoint. A *hypercone* is the union of a family of halfspaces that have a common edge.

In Chapter 10, the family of halfspaces of a linear space is converted into a geometrical system—called a *factor geometry*—by defining a join operation in it in a natural way. Factor geometries and join geometries share many common properties but differ markedly as algebraic systems, since a factor geometry has an identity element and its elements have inverses. The development has strong—though unforced—analogies with algebraic theories of congruence relations and factor or quotient systems.

Chapter 11 is devoted to the theory of *exchange geometries*, which are join geometries that satisfy a postulate equivalent to "two points determine a line." A theory of dimension is developed in an exchange geometry and the familiar incidence and intersection properties of lines and planes in Euclidean 3-space is generalized to finite-dimensional linear spaces.

Chapter 12 studies *ordered geometries*, which are join geometries that satisfy the Euclidean proposition: Of three distinct collinear points, one is between the other two. Among the results derived are basic geometric properties of polytopes; conditions for the separation of linear spaces by linear subspaces; the theorems of Radon, Helly, and Caratheodory on convex sets; and a striking formula for the linear space generated by a finite set of points.

In Chapter 13 various properties of polytopes in $\mathbb{R}^n$ are extended to ordered geometries. In particular, polytopes are related to intersections of halfspaces.

Since our approach is so different from the usual one, we felt compelled to develop the material slowly and deliberately with much concrete geometric motivation. This was done because of the unfamiliarity, not the inherent difficulty, of the treatment. The book assumes little formal knowledge of geometry and indeed little beyond high school algebra and some familiarity with intuitive set theory.

The book can be studied rather flexibly. Chapter 1 helps to provide a transition from intuitive informal geometry to an axiomatic formal development and can be read by able high school upperclassmen. The reader who has some degree of mathematical maturity need not begin with Chapter 1 but can use it as a source of supplementary material for the first few chapters. Chapters 2–6 form a basic course sequence in the abstract theory. (Some sections may be omitted in a first reading, for example: 2.20, 2.25, 2.26, 3.12–3.15, 4.20, 4.21, 4.23–4.26, and 6.20–6.24.) Except for the definition of join geometry, Chapter 5 can be skirted. But the reader is advised to make some contact with the models presented, since they shed so much light on the theory.

Here are some longer sequences with different emphases: Chapters 2–7; 2–6, 8; 2–6, 11; 2–6, 8, 9; 2–6, 8–10; 2–6, 8, 12; 2–8, 12, 13. A structure chart which indicates the interrelations of the chapters appears below. Footnotes to the titles of Chapters 7, 8, 10, and 12 provide more detail on the interrelations of the chapters.

The text is accompanied by a large and varied collection of exercises. They include simple exploratory exercises, verifying or testing a conjecture in a model, proofs that require only a few steps, difficult problems (the most difficult are indicated by an asterisk), and problems that involve extending the theory, labelled Projects.

Although the book was written as an undergraduate text, graduate students and mathematicians may find it of interest. There may be curiosity about a contemporary approach to the classical geometry which is our heritage from the Greeks. Those for whom geometry has little intuitive appeal may be attracted by the striking and unexpected analogies that appear between join geometries and algebraic structures, especially abelian groups. Specialists in the theory of convex sets may be interested both because of the broad vistas that seem to be opened up by the join theoretic axiomatization of the subject and the questions that arise on the extent to which the familiar theory of convexity in $\mathbb{R}^n$ can be extended to a join geometry, or to special types of join geometries.

Acknowledgments

The first acknowledgment is a personal one of Walter Prenowitz and concerns the origin of the book.

About ten years ago, Victor Klee suggested that I write a book on convex sets from the join theoretic point of view I had employed in an expository paper (Prenowitz [3]). I thought very well of the idea but felt the time was not ripe for so radical a departure in the treatment of geometry. The project was discussed when we met several times during the ensuing year and finally I changed my mind, feeling that Klee's initiative in proposing the project and persistence and courage in backing it were sufficient encouragement to take it on.

Prenowitz began by writing a preliminary version, which was published early in 1969 in a small soft-cover edition for classroom trial. Our thanks are extended to Jane W. DePaola, Branko Grünbaum, Walter Meyer, Paul T. Rygg, Seymour Schuster, and Ian Spatz for trying out this material. Also, we wish to thank William Barnier, George Booth, Melvin Hausner, Mark Hunacek, and Joseph Malkevitch for reading and criticizing portions of later versions of the manuscript.

Sophie Prenowitz was a mainstay during the whole period of evolution of the book from the preliminary version to the final manuscript. She typed, drew diagrams, proofread, and edited nontechnical portions of the manuscript. The cover drawing, which uses an oak leaf to illustrate the concept of a convex hull, was designed by her. She was a constant source of encouragment and support.

We are very grateful to the artist Saul Steinberg for permission to use his drawing as our frontispiece.

January 1979

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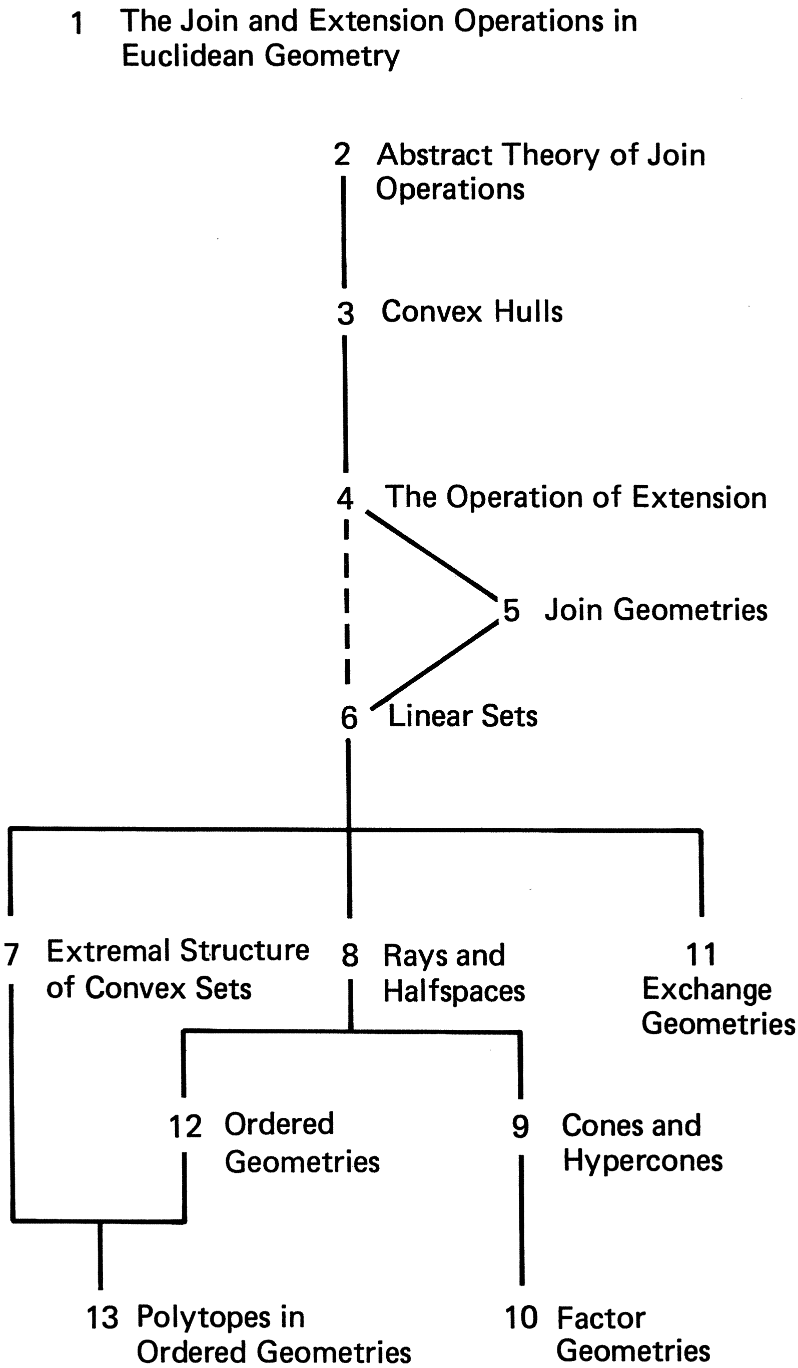

Note. Chapter 1 is not a prerequisite for the other chapters. The broken line indicates that it is possible to proceed directly from Chapter 4 to Chapter 6.