Extensions and Absolutes of Hausdorff Spaces

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To Peggy and Sheila

PREFACE

An **extension** of a topological space X is a space that contains X as a dense subspace. The construction of extensions of various sorts — compactifications, realcompactifications, H-closed extensions — has long been a major area of study in general topology. A ubiquitous method of constructing an extension of a space is to let the "new points" of the extension be ultrafilters on certain lattices associated with the space. Examples of such lattices are the lattice of open sets, the lattice of zero-sets, and the lattice of clopen sets.

A less well-known construction in general topology is the "absolute" of a space. Associated with each Hausdorff space X is an extremally disconnected zero-dimensional Hausdorff space EX, called the **Iliadis absolute** of X, and a perfect, irreducible, Θ-continuous surjection from EX onto X. A detailed discussion of the importance of the absolute in the study of topology and its applications appears at the beginning of Chapter 6. What concerns us here is that in most constructions of the absolute, the points of EX are certain ultrafilters on lattices associated with X. Thus extensions and absolutes, although conceptually very different, are constructed using similar tools.

The purpose of this book is to undertake a systematic study of the extensions and absolutes of Hausdorff spaces. A secondary theme is to show that by investigating the structure of certain lattices of subsets of a space, we obtain powerful tools that allow us to build many sorts of extensions and to construct absolutes in a variety of ways. Hence another purpose of this book is to demonstrate the power of lattice-theoretic concepts when applied to topology.

Chapter 1 is devoted to a discussion of some topics from elementary topology. In Chapter 2 we develop the basic concepts of lattices, filters, and convergence. We then discuss linearly ordered topological spaces, and finish the chapter with a discussion of ordinal and cardinal numbers. In Chapter 3 we discuss Boolean algebras and the Stone duality theorem. We finish by introducing Martin's axiom and some of its topological consequences.

Chapters 1 to 3 are preparatory material. In Chapters 4 to 7 we undertake a detailed study of the central matter of the book. Chapter 4 begins with a discussion of the general theory of extensions, and then embarks upon a detailed discussion of different methods of constructing compactifications. The chapter closes with a discussion of H-closed spaces. In Chapter 5 we investigate those topological properties P for which each "suitable" space has a "largest" extension with P. At the end there is a detailed discussion of realcompact spaces and extensions. Chapter 6 is devoted to the construction of the Iliadis and Banaschewski absolutes of a space. Chapter 7 studies the various types of H-closed extensions that a space may have, and discusses their interrelationships.

Chapter 8 is a collection of four essentially unrelated topics. After a brief introduction in 8.1, we investigate when the absolute of an extension is "the same" as the extension of the absolute. In 8.2 we are concerned with the commuting of absolutes with various H-closed extensions; in 8.3 we study when the Iliadis absolute commutes with extension properties containing realcompactness. In 8.4 we generalize the notion of absolutes and develop a theory of "covers" of spaces that is in many ways analogous to the theory of extensions discussed in Chapters 4 and 5. In 8.5 we relate real-valued continuous functions on a space to those on its absolute.

Chapter 9 provides a brief introduction to abstract category theory, and then concentrates on interpreting the construction of extensions and absolutes discussed in previous chapters in a category-theoretic light.

Each chapter is followed by a lengthy collection of problems. Some are routine verifications; others are harder. A number of hints are provided to aid the reader in solving the more difficult problems. The presence of these problems will, we hope, make this book useful as a text for a graduate course in topology. We have often referred in the body of the text to results appearing in problems from previous chapters, and we encourage the reader to attempt as many problems as possible.

We have not tried to give an exhaustive description of "who proved what," but some historical comments along these lines, together with guides to further reading, appear in the "Notes."

The heart of this book is Chapters 4 to 7, together with 8.4 (Chapter 9 is in part a retrospective, couched in the language of category theory, of what happened earlier). Readers with a good knowledge of topology who wish to reach the central ideas as soon as possible are advised to start at Chapter 4, and refer back to Chapters 1, 2, and 3 when necessary for notation, terminology, and basic results.

We do assume that the reader is familiar with the basic ideas of general topology. Some topics of particular relevance to our later work are reviewed in Chapter 1. The texts by Willard [Wi], Dugundji [Du], and Engelking [En] are excellent references for topological ideas not discussed here.

Although general topology in recent years has become increasingly dependent on axiomatic set theory, we do not assume that the reader has a detailed knowledge of this subject. It will suffice for the reader to know that the axioms of set theory can be formulated precisely in the language of mathematical logic. The formulation that we will implicitly use is the Zermelo-Frankel axioms, together with the Axiom of Choice (henceforth abbreviated ZFC). We do not state these axioms explicitly in this book. The reader who wishes to pursue these ideas is referred to the text by Kunen [Ku], which gives a rigorous treatment of those aspects of set theory most useful in topology.

We do assume that the reader is familiar with the distinction between a set and a proper class, with transfinite induction, and with the basic facts about ordinal and cardinal numbers. However, in 2.7 we give a rapid review (without proofs) of those facts about ordinal and cardinal numbers that are needed in this book.

This is not a book about set-theoretic topology, and we will not often be concerned with topological questions whose answers depend upon which "relatively consistent" model of ZFC is being assumed. Most of the theorems we discuss are "real" theorems in the sense that they are derivable from the axioms of ZFC, and do not require any additional set-theoretic assumptions for their proofs. There is one major exception to this, however; at the end of Chapter 3 we include a discussion of the continuum hypothesis and Martin's axiom. The reason for doing this is that our previous work on Boolean algebras and filters has equipped us to formulate the various versions of Martin's axiom, and prove their equivalence, with relatively little additional work. It also provides the reader with an opportunity to see one of the major tools of modern set-theoretic topology at work. Exercises using Martin's axiom and/or the continuum hypothesis are scattered throughout the problem sections of Chapter 3 and subsequent chapters.

Finally, we wish to acknowledge the extensive and valuable assistance we have received from our students, colleagues, and typists. Melvin Henriksen, Mohan Tikoo, and Johannes Vermeer read portions of an early version of the manuscript and provided us with many useful suggestions. A number of the problems reflect Vermeer's valuable contributions to the theory of H-closed spaces and absolutes. Beverly Diamond read Chapters 4 and 5, and the problems for Chapter 5, and made several valuable contributions to the final form of this portion of the book. Above all, we would like to express our sincere thanks and gratitude to Alan Dow, who read essentially the entire book to the end of Chapter 7, did virtually all of the problems, and protected us from numerous errors (in both style and substance). We appreciate the work done by our typists — Sharon Gumm, Carol Johnson, Susan Levine, Beverly Preiss, Everly Scherko, and Edith Despins — who had to endure our sloppy handwriting and numerous changes of mind. It has been a pleasure to work with such helpful and generous colleagues.

Needless to say, whatever errors remain are the sole responsibility of the authors.

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