

HANDBOOK OF

**EXACT
SOLUTIONS
for ORDINARY
DIFFERENTIAL
EQUATIONS**

SECOND EDITION

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 - 2.9.2-1. Argument of the arbitrary functions is
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 - 2.9.2-3. Other arguments of the arbitrary functions

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- 2.9.3. Equations of the Form $(y, x)'' + \sum_{i=0}^M (y, x)^{(i)} = 0$ ($M = 2, 3, 4$)
 - 2.9.3-1. Argument of the arbitrary functions is x .
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 - 2.9.3-3. Other arguments of arbitrary functions
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 - 3.1.2-1. Equations of the form $y''' + p_0(x)y = g(x)$
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 - 3.1.2-3. Equations of the form $y''' + p_2(x)y'' + p_1(x)y' + p_0(x)y = g(x)$
- 3.1.3. Equations Containing Exponential Functions
 - 3.1.3-1. Equations with exponential functions
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- 3.1.4. Equations Containing Hyperbolic Functions
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- 3.1.7. Equations Containing Inverse Trigonometric Functions
- 3.1.8. Equations Containing Combinations of Exponential, Logarithmic, Trigonometric, and Other Functions
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 - 3.1.9-1. Equations of the form $y''' + p_1(x)y' + p_0(x)y = g(x)$
 - 3.1.9-2. Equations of the form $y''' + p_2(x)y'' + p_1(x)y' + p_0(x)y = g(x)$

3.2. Equations of the Form $y''' = A(x)(y')^m(y'')$

- 3.2.1. Classification Table
- 3.2.2. Equations of the Form $y''' = A(x)y''$

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- 3.2.3. Equations of the Form $y''' = A$
 - 3.2.4. Equations with $|y| + |y'| \neq 0$
 - 3.2.5. Some Transformations
 - 3.3. Equations of the Form $y''' = f(x)g(y')(y'')$
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 - 3.3.2. Equations Containing Exponential Functions
 - 3.3.3. Other Equations
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 - 3.4.1. Equations Containing Power Functions
 - 3.4.1-1. Equations of the form $(y, y')''' = g(y, y')$
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 - 3.4.1-4. Other equations
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 - 3.4.2-1. Equations of the form $y''' = f(x, y, y')$
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 - 3.4.3-1. Equations with hyperbolic sine
 - 3.4.3-2. Equations with hyperbolic cosine
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 - 3.4.4-1. Equations of the form $y''' = f(x, y, y')$
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 - 3.4.5-1. Equations with sine
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 - 3.4.5-3. Equations with tangent
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 - 3.5.2-1. Arguments of the arbitrary functions depend on x and y
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 - 3.5.3-1. The arbitrary functions depend on x or y
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4. Fourth-Order Differential Equations

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4.1.1. Preliminary Remarks

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4.1.2-2. Equations of the form $y^{(4)} + p_1(x)y' + p_0(x)y = g(x)$

4.1.2-3. Equations of the form $y^{(4)} + p_2(x)y'' + p_1(x)y' + p_0(x)y = g(x)$

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4.1.3. Equations Containing Exponential and Hyperbolic Functions

4.1.3-1. Equations with exponential functions

4.1.3-2. Equations with hyperbolic functions

4.1.4. Equations Containing Logarithmic Functions

4.1.5. Equations Containing Trigonometric Functions

4.1.5-1. Equations with sine and cosine

4.1.5-2. Equations with tangent and cotangent

4.1.6. Equations Containing Arbitrary Functions

4.1.6-1. Equations of the form $y^{(4)} + p_1(x)y' + p_0(x)y = g(x)$

4.1.6-2. Equations of the form $y^{(4)} + p_2(x)y'' + p_1(x)y' + p_0(x)y = g(x)$

4.1.6-3. Other equations

4.2. Nonlinear Equations

4.2.1. Equations Containing Power Functions

4.2.1-1. Equations of the form $y^{(4)} = f(x, y)$

4.2.1-2. Equations of the form $y^{(4)} = f(x, y, y')$

4.2.1-3. Equations of the form $y^{(4)} = f(x, y, y', y'')$

4.2.1-4. Equations of the form $y^{(4)} = f(x, y, y', y'', y''')$

4.2.2. Equations Containing Exponential Functions

4.2.2-1. Equations of the form $y^{(4)} = f(x, y)$

4.2.2-2. Other equations

4.2.3. Equations Containing Hyperbolic Functions

4.2.3-1. Equations with hyperbolic sine

4.2.3-2. Equations with hyperbolic cosine

4.2.3-3. Equations with hyperbolic tangent

4.2.3-4. Equations with hyperbolic cotangent

4.2.4. Equations Containing Logarithmic Functions

4.2.4-1. Equations of the form $y^{(4)} = f(x, y)$

4.2.4-2. Other equations

4.2.5. Equations Containing Trigonometric Functions

4.2.5-1. Equations with sine

4.2.5-2. Equations with cosine

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4.2.5-4. Equations with cotangent

4.2.6. Equations Containing Arbitrary Functions

4.2.6-1. Equations of the form $y^{(4)} = f(x, y)$

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4.2.6-3. Equations of the form $y^{(4)} = f(x, y, y', y'')$

4.2.6-4. Equations of the form $y^{(4)} = f(x, y, y', y'', y''')$

4.2.6-5. Other equations

5. Higher-Order Differential Equations

5.1. Linear Equations

5.1.1. Preliminary Remarks

5.1.2. Equations Containing Power Functions

5.1.2-1. Equations of the form $y^{(n)} + p_0(x)y = g(x)$

5.1.2-2. Equations of the form $y^{(n)} + p_1(x)y' + p_0(x)y = g(x)$

5.1.2-3. Other equations

5.1.3. Equations Containing Exponential and Hyperbolic Functions

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5.1.3-2. Equations with hyperbolic functions

5.1.4. Equations Containing Logarithmic Functions

5.1.5. Equations Containing Trigonometric Functions

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5.1.6. Equations Containing Arbitrary Functions

5.1.6-1. Equations of the form $y^{(n)} + p_1(x)y' + p_0(x)y = g(x)$

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5.2.1-4. Other equations

5.2.2. Equations Containing Exponential Functions

5.2.2-1. Fifth- and sixth-order equations

5.2.2-2. Equations of the form $y^{(n)} = f(x, y)$

5.2.2-3. Other equations

5.2.3. Equations Containing Hyperbolic Functions

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5.2.3-3. Equations with hyperbolic tangent

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5.2.4. Equations Containing Logarithmic Functions

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5.2.5. Equations Containing Trigonometric Functions

5.2.5-1. Equations with sine

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5.2.6. Equations Containing Arbitrary Functions

5.2.6-1. Fifth- and sixth-order equations

5.2.6-2. Equations of the form $y^{(n)} = f(x, y)$

5.2.6-3. Equations of the form $y^{(n)} = f(x, y, y')$

5.2.6-4. Equations of the form $y^{(n)} = f(x, y, y', y'')$

5.2.6-5. Equations of the form $(y^{(n)})^{(k)} + g(x, y, y', y'')^{(-1)} = f(x, y, y', y'')^{(-2)}$

5.2.6-6. Equations of the form $y^{(n)} = f(x, y, y', y'')^{(-1)}$

5.2.6-7. Equations of the general form $(y^{(n)})^{(k)} = f(x, y, y', y'')^{(-1)}$

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- S.1.1-2. Relations between trigonometric functions of single argument
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