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Problems and Theorems in Analysis II

Theory of Functions. Zeros.
Polynomials. Determinants.
Number Theory. Geometry

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