

Texts in Applied Mathematics **9**

Editors

F. John

J.E. Marsden

L. Sirovich

M. Golubitsky

W. Jäger

Texts in Applied Mathematics

1. *Sirovich*: Introduction to Applied Mathematics.
2. *Wiggins*: Introduction to Applied Nonlinear Dynamical Systems and Chaos.
3. *Hale/Koçak*: Dynamics and Bifurcations.
4. *Chorin/Marsden*: A Mathematical Introduction to Fluid Mechanics, 2nd ed.
5. *Hubbard/West*: Differential Equations: A Dynamical Systems Approach, Part I: Ordinary Differential Equations.
6. *Sontag*: Mathematical Control Theory: Deterministic Finite Dimensional Systems.
7. *Perko*: Differential Equations and Dynamical Systems.
8. *Seaborn*: Hypergeometric Functions and Their Applications.
9. *Pipkin*: A Course on Integral Equations.
10. *Hoppensteadt/Peskin*: Mathematics in Medicine and the Life Sciences.

Allen C. Pipkin

A Course on Integral Equations



Springer Science+Business Media, LLC

Allen C. Pipkin
Center of Fluid Mechanics, Turbulence
and Computation
Brown University
Providence, Rhode Island 02912
USA

Editors

F. John
Courant Institute of
Mathematical Sciences
New York University
New York, NY 10012
USA

J.E. Marsden
Department of
Mathematics
University of California
Berkeley, CA 94720
USA

L. Sirovich
Division of Applied
Mathematics
Brown University
Providence, RI 02912
USA

M. Golubitsky
Department of
Mathematics
University of Houston
Houston, TX 77004
USA

W. Jäger
Department of Applied
Mathematics
Universität Heidelberg
Im Neuenheimer Feld 294
6900 Heidelberg, FRG

Mathematics Subject Classification: 45-01, 45B05, 45E05, 45E10

With 8 illustrations

Library of Congress Cataloging-in-Publication Data
Pipkin, A. C.

A course on integral equations / by Allen C. Pipkin.

p. cm.

Includes bibliographical references and index.

1. Integral equations I. Title.

QA431.P568 1991

515'.45—dc20

91-27827

CIP

Printed on acid-free paper

© 1991 Springer Science+Business Media New York
Originally published by Springer-Verlag New York, Inc. in 1991
Softcover reprint of the hardcover 1st edition 1991

All rights reserved. This work may not be translated or copied in whole or in part without the written permission of the publisher Springer Science+Business Media, LLC, except for brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed is forbidden.

The use of general descriptive names, trade names, trademarks, etc., in this publication, even if the former are not especially identified, is not to be taken as a sign that such names, as understood by the Trade Marks and Merchandise Marks Act, may accordingly be used freely by anyone.

Photocomposed copy prepared using LaTeX.

9 8 7 6 5 4 3 2 1

ISBN 978-1-4612-8773-5 ISBN 978-1-4612-4446-2 (eBook)

DOI 10.1007/978-1-4612-4446-2

Series Preface

Mathematics is playing an ever more important role in the physical and biological sciences, provoking a blurring of boundaries between scientific disciplines and a resurgence of interest in the modern as well as the classical techniques of applied mathematics. This renewal of interest, both in research and teaching, has led to the establishment of the series: *Texts in Applied Mathematics (TAM)*.

The development of new courses is a natural consequence of a high level of excitement on the research frontier as newer techniques, such as numerical and symbolic computer systems, dynamical systems, and chaos, mix with and reinforce the traditional methods of applied mathematics. Thus, the purpose of this textbook series is to meet the current and future needs of these advances and encourage the teaching of new courses.

TAM will publish textbooks suitable for use in advanced undergraduate and beginning graduate courses, and will complement the *Applied Mathematical Sciences (AMS)* series, which will focus on advanced textbooks and research level monographs.

Foreword

This book is based on a one-semester course for graduate students in the physical sciences and applied mathematics. No great mathematical background is needed, but the student should be familiar with the theory of analytic functions of a complex variable. Since the course is on problem-solving rather than theorem-proving, the main requirement is that the student should be willing to work out a large number of specific examples.

The course is divided about equally into three parts. The first part, on Fredholm and Hilbert–Schmidt theory, is mostly theoretical. The last two parts emphasize problem-solving, using complex variables. The solution of convolution equations by using one- and two-sided Laplace transforms is covered in the second part, and the third part deals with Cauchy principal value integrals.

The chapters on Fredholm theory and Hilbert–Schmidt theory stress the analogy with linear algebra, since that is the point of these subjects, and much of the discussion is presented in terms of finite-dimensional matrices and vectors. Students who understand these chapters should be able to read the more detailed treatment given by Riesz and Sz.-Nagy without getting lost.

In preparation for the material on convolution equations, there is a short introduction to the Laplace transform. For reference, this chapter also includes proofs of some of the theorems that are taken for granted in lectures. Many of these proofs are adapted from Widder's book on Laplace transforms.

The character of Laplace transforms as analytic functions is emphasized in the chapters on convolution equations. The use of analytic function theory continues in the final part of the course, which begins with a chapter on the evaluation of principal value integrals. Again for reference, this chapter also includes proofs of some of the theorems that are being used. Widder's book on integral transforms is a good source for such theorems.

The principal value equations that we consider always have some or all of the real axis as the integration contour. Understanding this case seems to be sufficient for understanding the results about arbitrary contours in Muskhelishvili's treatise, and in fact we can go a little further than Muskhelishvili in the matter of infinite contours.

The complex variable methods that are used in the last two-thirds of the course put the student in a position to understand solutions based on the Wiener–Hopf technique. There is only a relatively short chapter on Wiener–Hopf equations, giving a few examples in which the analysis is completely concrete.

I would like to thank Larry Sirovich for his encouragement in this project. Kate MacDougall deserves particular thanks for her skill and patience in preparing the manuscript.

Allen C. Pipkin

References

- F. Riesz and B. Sz.-Nagy, *Functional Analysis*, Ungar, New York, 1955.
D.V. Widder, *The Laplace Transform*, University Press, Princeton, 1941.
D.V. Widder, *An Introduction to Transform Theory*, Academic Press, New York, 1971.
N.I. Muskhelishvili, *Singular Integral Equations*, Wolters-Noordhoff, Groningen, 1958.

Contents

Series Preface	v
Foreword	vii
1 Fredholm Theory	1
1.1 Fredholm Equations. Notation	1
1.2 Purpose of Fredholm Theory	3
1.3 A Separable Equation	4
1.4 Separable Kernels	5
1.5 Resolvent Kernel	6
1.6 Approximation by Separable Kernels	7
1.7 Uniqueness Implies Existence	10
1.8 Fredholm Conditions	12
1.9 The Fredholm Alternative	14
1.10 Eigenfunctions	14
1.11 Schmidt's Method	16
1.12 Iteration. The Neumann Series	17
1.13 Norms	19
1.14 Convergence	21
1.15 The Neumann Series for an Integral Equation	22
1.16 Existence and Uniqueness for Small Kernels	24
1.17 Fredholm Equations	25
1.18 Singularities of the Resolvent Kernel	26
2 Fredholm Theory with Integral Norms	28
2.1 Absolutely Integrable Functions	28
2.2 The L^1 Norm of an Operator	30
2.3 Fredholm Theory in L^1	30
2.4 The Euclidean Norm	31
2.5 L^2 Theory	32
2.6 Approximation by Orthogonal Functions	33
2.7 Completeness	34

2.8	Approximation by Separable Kernel	36
2.9	Existence of Eigenvectors	38
2.10	Weak Convergence	39
2.11	Infinite Matrices	40
2.12	Existence of Eigenfunctions	42
2.13	L^p Theory	42
3	Hilbert–Schmidt Theory	44
3.1	Eigenvalues and Eigenvectors	44
3.2	Spectral Representation	45
3.3	Example	47
3.4	Resolvent Kernel	48
3.5	Circulants	48
3.6	Real Symmetric Matrices	51
3.7	The Euclidean Norm of a Matrix	54
3.8	The Rayleigh Quotient	57
3.9	Hilbert–Schmidt Theory	59
3.10	The Hilbert–Schmidt Theorem	61
3.11	Weighted Inner Products	62
3.12	Example: Fiber-Reinforced Materials	63
3.13	Green’s Functions	66
3.14	Example: Hanging String	68
3.15	Sturm–Liouville Problems	69
3.16	Eigenfunction Expansions	69
4	Laplace Transforms	72
4.1	Fourier Transforms	72
4.2	One-Sided Laplace Transforms	75
4.3	Analyticity	75
4.4	Convergence	76
4.5	Examples and Formulas	78
4.6	Convergence Proofs	79
4.7	The Factorial Function	81
4.8	Transforms of Powers and Logs	82
4.9	Convolutions	82
4.10	Transforms of Integrals	83
4.11	Transforms of Derivatives	84
4.12	Derivatives of Transforms	85
4.13	Integrals of Transforms	86
4.14	The Inversion Integral	87
4.15	Loop Integrals	88
4.16	Watson’s Lemma for Loop Integrals	91

5	Volterra Equations	92
5.1	Volterra Equations	92
5.2	Uniqueness	93
5.3	Equations of the First Kind	96
5.4	Convolutions	97
5.5	Fractional Integration	98
5.6	Fractional Integral Equations	100
5.7	Example: Fiber-Bound Pressure Vessels	101
5.8	Translation Invariance	103
5.9	Transforms of Convolutions	104
5.10	Transforms of One-Sided Functions	105
6	Reciprocal Kernels	107
6.1	Linear Input-Output Relations	107
6.2	Examples	109
6.3	Inversion of Input-Output Relations	110
6.4	Solution by Laplace Transforms	111
6.5	Transforms of Reciprocal Kernels	112
6.6	Dissipative Systems	113
6.7	Average Values	114
6.8	Averages of Reciprocal Kernels	116
6.9	Initial Behavior	117
6.10	Regularly-Varying Functions	119
6.11	Barely Integrable Kernels	121
6.12	Existence of the Reciprocal Kernel	122
6.13	Long-Time Behavior	124
6.14	Loop Integrals	125
6.15	Long-Time Behavior. Examples	128
6.16	Reciprocals of Positive Decreasing Kernels	130
6.17	Completely Passive Systems	131
6.18	Completely Monotone Kernels	133
7	Smoothing and Unsmoothing	137
7.1	Moving Averages	137
7.2	Example: Lateral Inhibition in Vision	140
7.3	Two-Sided Laplace Transforms	141
7.4	The Inversion Integral	143
7.5	A One-Sided Averager	144
7.6	Non-Uniqueness	146
7.7	Generalization of the Solution	147
7.8	Stability and Uniqueness for One-Sided Averagers	148

7.9	Another One-Sided Averager	151
7.10	The Picard Equation	153
7.11	Symmetric Averagers	155
7.12	Example: Reciprocal Cosh	156
7.13	Disguised Convolutions	158
8	Wiener–Hopf Equations	160
8.1	Wiener–Hopf Equations	160
8.2	An Example	161
8.3	Eigenfunction	164
8.4	Equations of the First Kind	165
8.5	Example	166
8.6	Separations	167
8.7	Wiener–Hopf with Rational Transforms	168
9	Evaluation of Principal Value Integrals	173
9.1	Cauchy Kernel	173
9.2	Plemelj Formulas for Analytic Density	175
9.3	Evaluation of Principal Value Integrals. Example	176
9.4	Splitting	177
9.5	The Imaginary Part	180
9.6	An Integral over a Finite Interval	183
9.7	Thin Airfoils	185
9.8	A Density with an Essential Singularity	186
9.9	Non-Integrable Singularity	187
9.10	Upper and Lower Half-Plane Functions	189
9.11	Convergence	191
9.12	Behavior at Infinity. Finite Intervals	193
9.13	Infinite Intervals	193
9.14	Behavior at Infinity. Infinite Intervals	195
9.15	Identification of Upper Half-Plane Functions	196
9.16	Plemelj Formulas I	198
9.17	Plemelj Formulas II	200
9.18	Hölder Continuity	201
10	Cauchy Principal Value Equations on a Finite Interval	203
10.1	Boundary Value Problems	203
10.2	The Homogeneous Equation	205
10.3	Equations of the First Kind	207
10.4	Example 1	208

Contents		xiii
10.5	Example 2	210
10.6	Eigenfunctions	212
10.7	Equations of the Second Kind	213
10.8	Example	216
11	Principal Value Equations on a Semi-Infinite Interval	217
11.1	Equations on a Semi-Infinite Interval	217
11.2	Equations of the First Kind on $(0, \infty)$	218
11.3	Example 1	219
11.4	Example 2	220
11.5	Example 3	222
11.6	Superposition	222
11.7	Equations of the Second Kind on $(0, \infty)$	225
11.8	Example 1	226
11.9	Example 2	227
11.10	Homogeneous Equations of the Third Kind	228
11.11	Step Functions	229
11.12	Some Cases with $p(x)$ Continuous	231
11.13	A Barely Integrable Solution	233
12	Principal Value Equations on an Infinite Interval	235
12.1	Equations on the Whole Line	235
12.2	Equations of the First Kind	236
12.3	Gentlemen's Theorem No. 1	239
12.4	Equations of the First Kind: Examples	240
12.5	Example 1	241
12.6	Example 2	242
12.7	Example 3	243
12.8	Example 4	244
12.9	Eigenfunctions for the Hilbert Transform	246
12.10	Equations of the Second Kind on the Whole Line	247
12.11	Homogeneous Equations of the Third Kind	249
12.12	Step Functions	250
12.13	Continuous Cases	251
12.14	Inhomogeneous Equations of the Third Kind	253
12.15	Example	255
	Solutions of Selected Problems	257
	Index	265