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Convex Functions, Monotone Operators and Differentiability

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PREFACE

In the three and a half years since the first edition to these notes was written there has been progress on a number of relevant topics. D. Preiss answered in the affirmative the decades old question of whether a Banach space with an equivalent Gâteaux differentiable norm is a weak Asplund space, while R. Haydon constructed some very ingenious examples which show, among other things, that the converse to Preiss' theorem is false. S. Simons produced a startlingly simple proof of Rockafellar's maximal monotonicity theorem for subdifferentials of convex functions. G. Godefroy, R. Deville and V. Zizler proved an exciting new version of the Borwein-Preiss smooth variational principle. Other new contributions to the area have come from J. Borwein, S. Fitzpatrick, P. Kenderov, I. Namioka, N. Ribarska, A. and M. E. Verona and the author.

Some of the new material and substantial portions of the first edition were used in a one-quarter graduate course at the University of Washington in 1991 (leading to a number of corrections and improvements) and some of the new theorems were presented in the Rainwater Seminar. An obvious improvement is due to the fact that I learned to use TEX. The task of converting the original MacWrite text to TEXwas performed by Ms. Mary Sheetz, to whom I am extremely grateful.

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Robert R. Phelps
February 6, 1992
Seattle, Washington

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PREFACE TO THE FIRST EDITION

These notes had their genesis in a widely distributed but unpublished set of notes *Differentiability of convex functions on Banach spaces* which I wrote in 1977–78 for a graduate course at University College London (UCL). Those notes were largely incorporated into J. Giles' 1982 Pitman Lecture Notes *Convex analysis with application to differentiation of convex functions*. In the course of doing so, he reorganized the material somewhat and took advantage of any simpler proofs available at that time. I have not hesitated to return the compliment by using a few of those improvements. At my invitation, R. Bourgin has also incorporated material from the UCL notes in his extremely comprehensive 1983 Springer Lecture Notes *Geometric aspects of convex sets with the Radon-Nikodym property*. The present notes do not overlap too greatly with theirs, partly because of a substantially changed emphasis and partly because I am able to use results or proofs that have come to light since 1983.

Except for some subsequent revisions and modest additions, this material was covered in a graduate course at the University of Washington in Winter Quarter of 1988. The students in my class all had a good background in functional analysis, but there is not a great deal needed to read these notes, since they are largely self-contained; in particular, no background in convex functions is required. The main tool is the separation theorem (a.k.a. the Hahn-Banach theorem); like the standard advice given in mountaineering classes (concerning the all-important bowline for tying oneself into the end of the climbing rope), you should be able to employ it using only one hand while standing blindfolded in a cold shower.

These notes have been influenced very considerably by frequent conversations with Isaac Namioka (who has an almost notorious instinct for simplifying proofs) as well as occasional conversations with Terry Rockafellar; I am grateful to them both. I am also grateful to Jon Borwein, Marian Fabian and Simon Fitzpatrick, each of whom sent me useful suggestions based on a preliminary version.

Robert R. Phelps
October 5, 1988
Seattle, Washington

INTRODUCTION

The study of the differentiability properties of convex functions on infinite dimensional spaces has continued on and off for over fifty years. There are a couple of obvious reasons for this. Aside from the intrinsic interest of investigating the many consequences implicit in something as simple as convexity, there is the satisfaction (for this author, at least) in discovering that a number of apparently disparate mathematical topics (extreme points – rather, strongly exposed points – of noncompact convex sets, monotone operators, perturbed optimization of real-valued functions, differentiability of vector-valued measures) are in fact closely intertwined, with differentiability of convex functions forming a common thread.

Starting in Section 1 with the definition of convex functions and a fundamental differentiability property in the one-dimensional case [right-hand and left-hand derivatives always exist], we get quickly to the first infinite dimensional result, Mazur's intriguing 1933 theorem: A continuous convex function on a separable Banach space has a dense G_δ set of points where it is Gâteaux differentiable. In order to go beyond Mazur's theorem, some time is spent in studying the *subdifferential* of a convex function f; this is a set-valued map from the space to its dual whose image at each point x consists of all plausible candidates for the derivative of f at x. [The function f is Gâteaux differentiable precisely when the subdifferential is single-valued, and it is Fréchet differentiable precisely when its subdifferential is single-valued and norm-to-norm continuous.]

Since a subdifferential is a special case of a *monotone operator*, Section 2 starts with a detailed look at monotone operators. These objects are of independent origin, having been extensively studied in the sixties and early seventies by numerous mathematicians (with major contributions from H. Brezis, F. Browder and G. J. Minty) in connection with nonlinear partial differential equations and other aspects of nonlinear analysis. (See, for instance, [Bre], [De], [Pa-Sb] or [Ze]). Also in the sixties, an in-depth study of monotone operators in fairly general spaces was carried out by R. T. Rockafellar, who established a number of fundamental properties, such as their local boundedness. He also gave an elegant characterization of those monotone operators which are the subdifferentials of convex functions. [The connection between monotone operators and derivatives of convex functions is readily apparent on the real line, since single-valued monotone operators coincide in that case with monotone nondecreasing functions, as do the right-hand derivatives of convex functions of one variable.]

Mazur's theorem was revisited 30 years later by J. Lindenstrauss, who showed in 1963 that if a separable Banach space is assumed to be reflexive, then Mazur's conclusion about Gâteaux differentiability could be strengthened to Fréchet differentiability. In 1968, E. Asplund extended Mazur's theorem in two ways: He found more general spaces in which the same conclusion holds, and he studied a smaller class of spaces (now called Asplund spaces) in which Lindenstrauss' Fréchet differentiability conclusion is valid. Asplund used an

ingenious combination of analytic and geometric techniques to prove some of
the basic theorems in the subject. Roughly ten years later, P. Kenderov (as well
as R. Robert and S. Fitzpatrick) proved some general continuity theorems for
monotone operators which, when applied to subdifferentials, yield Asplund's
results as special cases. In Section 2 we follow this approach, incorporating
recent work by D. Preiss and L. Zajicek to obtain the major differentiability
theorems.

The results of Section 2 all involve *continuous* convex functions defined
on open convex sets. For many applications, it is more suitable to consider
lower semicontinuous convex functions, even those which are extended real
valued (possibly equal to $+\infty$). (For instance, in many optimization problems
one finds just such a function in the form of the supremum of an infinite fam-
ily of affine continuous functions.) Lower semicontinuous convex functions also
yield a natural way to translate results about closed convex sets into results
about convex functions and vice versa. (For instance, the set of points on or
above the graph of such a convex function – its *epigraph* – forms a closed con-
vex set). In Section 3 one will find some classical results (various versions and
extensions of the Bishop-Phelps theorems) which, among other things, guaran-
tee that subdifferentials still exist for lower semicontinuous convex functions.
A nonconvex version of this type of theorem is I. Ekeland's *variational prin-
ciple*, which asserts that a lower semicontinuous function which nearly attains
its minimum at a point x admits arbitrarily small perturbations (by trans-
lates of the norm) which *do* attain a minimum, at points near x. This result,
while simple to state and prove, has been shown by Ekeland [Ek] to have
an extraordinarily wide variety of applications, in areas such as optimization,
mathematical programming, control theory, nonlinear semigroups and global
analysis.

In Section 4, we examine variational principles which use *differentiable*
perturbations. The first such result was due to J. Borwein and D. Preiss;
subsequently, this was recast in a different and somewhat simpler form by
G. Godefroy, R. Deville and V. Zizler; we follow their approach. Some
deep theorems about differentiability of convex functions fall out as fairly
easy corollaries, and it is reasonable to expect future useful applications. This
is followed by the generalization (to maximal monotone operators) of Preiss'
theorem that Gâteaux differentiability of the norm forces *every* continuous
convex function to be generically Gâteaux differentiable.

Section 5 describes the duality between Asplund spaces and spaces with
the *Radon-Nikodym property* (RNP). These are Banach spaces for which a
Radon-Nikodym-type differentiation theorem is valid for vector measures with
values in the space. Spaces with the RNP have an interesting history, starting
in the late sixties with the introduction by M. Rieffel of a geometric property
(*dentability*) which turned out to characterize the RNP and which has led
to a number of other characterizations in terms of the extreme points (or
strongly exposed points) of bounded closed convex subsets of the space. A truly
beautiful result in this area is the fact that a Banach space is an Asplund space
if and only if its dual has the RNP. (Superb expositions of the RNP may be

found in the books by J. Diestel and J. J. Uhl [Di-U] and R. Bourgin [Bou].)
In Section 5, the RNP is defined in terms of dentability, and a number of
basic results are obtained using more recent (and simpler) proofs than are
used in the above monographs. One will also find there J. Bourgain's proof
of C. Stegall's perturbed optimization theorem for semicontinuous functions
on spaces with the RNP; this yields as a corollary the theorem that in such
spaces every bounded closed convex set is the closed convex hull of its strongly
exposed points.

The notion of perturbed optimization has been moving closer to center
stage, since it not only provides a more general format for stating previously
known theorems, but also permits the formulation of more general results. The
idea is simple: One starts with a real-valued function f which is, say, lower
semicontinuous and bounded below on a nice set, and then shows that there
exist arbitrarily small perturbations g such that $f + g$ attains a minimum
on the set. The perturbations g might be restrictions of continuous linear
functionals of small norm, or perhaps Lipschitz functions of small Lipschitz
norm. Moreover, for really nice sets, the perturbed function attains a *strong*
minimum: Every mimimizing sequence converges.

The brief Section 6 is devoted to the class of Banach spaces in which every
continuous convex function is Gâteaux differentiable in a dense set of points
(dropping the previous condition that the set need be a G_δ). Some evidence
is presented that this is perhaps the "right" class to study.

Even more general than monotone operators is a class of set valued maps
(from a metric space, say, to a dual Banach space) which are upper semicon-
tinuous and take on weak* compact convex values, the so-called *usco* maps. In
Section 7, some interesting connections between monotone operators and usco
maps are described, culminating in a topological proof of one of P. Kenderov's
continuity theorems.

CONTENTS