Chaos and Fractals

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New Frontiers of Science

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To
Karin, Iris, and Gerlinde

Preface

Over the last decade, physicists, biologists, astronomers and economists have created a new way of understanding the growth of complexity in nature. This new science, called chaos, offers a way of seeing order and pattern where formerly only the random, erratic, the unpredictable — in short, the chaotic — had been observed.

James Gleick

This book is written for everyone who, even without much knowledge of technical mathematics, wants to know *the details* of chaos theory and fractal geometry. This is not a textbook in the usual sense of the word, nor is it written in a 'popular scientific' style. Rather, it has been our desire to give the reader a broad view of the underlying notions behind fractals, chaos and dynamics. In addition, we have wanted to show how fractals and chaos relate to each other and to many other aspects of mathematics as well as to natural phenomena. A third motif in the book is the inherent visual and imaginative beauty in the structures and shapes of fractals and chaos.

For almost ten years now mathematics and the natural sciences have been riding a wave which, in its power, creativity and expanse, has become an interdisciplinary experience of the first order. For some time now this wave has also been touching distant shores far beyond the sciences. Never before have mathematical insights — usually seen as dry and dusty — found such rapid acceptance and generated so much excitement in the public mind. Fractals and chaos have literally captured the attention, enthusiasm and interest of a world-wide public. To the casual observer, the color of their essential structures and their beauty and geometric form captivate the visual senses as few other things they have ever experienced in mathematics. To the student, they bring mathematics out of the realm of ancient history into the twenty-first century. And to the scientist, fractals and chaos offer a rich environment for exploring and modelling the complexity of nature.

But what are the reasons for this fascination? First of all, this young area of research has created pictures of such power and singularity that a collection of them, for example, has proven to be one of the most successful world-wide series of exhibitions ever sponsored by the Goethe-Institute.[2] More important, however, is the fact that chaos theory and fractal geometry have corrected an outmoded conception of the world.

The magnificent successes in the fields of the natural sciences and technology had, for many, fed the illusion that the world on the whole functioned like a huge clockwork mechanism, whose laws were only waiting to be deciphered step by step. Once the laws were known, it was believed, the evolution or development of things could — at least in principle — be ever more accurately predicted. Captivated by the breathtaking advances in the development of computer technology and its promises of a greater command of information, many have put increasing hope in these machines.

But today it is exactly those at the active core of modern science who are proclaiming that this hope is unjustified; the ability to see ever more accurately into future developments is unattainable.

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[1] J. Gleick, *Chaos - Making a New Science*, Viking, New York, 1987.

[2] Alone at the venerable London Museum of Science, the exhibition *Frontiers of Chaos: Images of Complex Dynamical Systems* by H. Jürgens, H.-O. Peitgen, M. Prüfer, P. H. Richter and D. Saupe attracted more than 140,000 visitors. Since 1985 this exhibition has travelled to more than 100 cities in more than 30 countries on all five continents.

One conclusion that can be drawn from the new theories, which are admittedly still young, is that stricter determinism and apparently accidental development are *not* mutually exclusive, but rather that their coexistence is more the rule in nature. Chaos theory and fractal geometry address this issue. When we examine the development of a process over a period of time, we speak in terms used in chaos theory. When we are more interested in the structural forms which a chaotic process leaves in its wake, then we use the terminology of fractal geometry, which is really the geometry whose structures are what give order to chaos.

In some sense, fractal geometry is first and foremost a new 'language' used to describe, model and analyze the complex forms found in nature. But while the elements of the 'traditional language' — the familiar Euclidean geometry — are basic visible forms such as lines, circles and spheres, those of the new language do not lend themselves to direct observation. They are, namely, algorithms, which can be transformed into shapes and structures only with the help of computers. In addition, the supply of these algorithmic elements is inexhaustibly large; and they are capable of providing us with a powerful descriptive tool. Once this new language has been mastered, we can describe the form of a cloud as easily and precisely as an architect can describe a house using the language of traditional geometry.

The correlation of chaos and geometry is anything but coincidental. Rather, it is a witness to their deep kinship. This kinship can best be seen in the Mandelbrot set, a mathematical object discovered by Benoit Mandelbrot in 1980. It has been described by some scientists as the most complex — and possibly the most beautiful — object ever seen in mathematics. Its most fascinating characteristic, however, has only just recently been discovered: namely, that it can be interpreted as an illustrated encyclopedia of an infinite number of algorithms. It is a fantastically efficiently organized storehouse of images, and as such it is *the* example par excellence of order in chaos.

Fractals and modern chaos theory are also linked by the fact that many of the contemporary pace-setting discoveries in their fields were only possible using computers. From the perspective of our inherited understanding of mathematics, this is a challenge which is felt by some to be a powerful renewal and liberation and by others to be a degeneration. However this dispute over the 'right' mathematics is decided, it is already clear that the history of the sciences has been enriched by an indispensable chapter. Only superficially is the issue one of beautiful pictures or of perils of deterministic laws. In essence, chaos theory and fractal geometry radically question our understanding of equilibria — and therefore of harmony and order — in nature as well as in other contexts. They offer a new holistic and integral model which can encompass a part of the true complexity of nature for the first time. It is highly probable that the new methods and terminologies will allow us, for example, a much more adequate understanding of ecology and climatic developments, and thus they could contribute to our more effectively tackling our gigantic global problems.

We have worked hard in trying to reveal the elements of fractals, chaos and dynamics in a non-threatening fashion. Each chapter can stand on its own and can be read independently from the others. Each chapter is centered around a running 'story' typeset in *Times* and printed toward the outer margins. More technical discussions, typeset in *Helvetica* and printed toward the inner margins, have been included to occasionally enrich the discussion by providing deeper analyses for those who may desire them and those who are prepared to work themselves through some mathematical notations. At the end of each chapter we offer a short BASIC program, the *Program of the Chapter*, which is designed to highlight one of the most prominent experiments of the respective chapter.

This book is a close relative of the two-volume set *Fractals for the Classroom* which was published by Springer-Verlag and the National Council of Teachers of Mathematics in 1991 and 1992. While those books were originally written for an audience which is involved with the teaching or learning of mathematics, this book is intended for a much larger readership. It combines most parts of the afore-mentioned books with many extensions and two important appendices.

The first appendix, written by Yuval Fisher, deals with aspects of *image compression* using fundamental ideas from fractal geometry. Such applications have been discussed for about five years and hopes of new breakthrough technologies have risen very high through the work and announcements of the group around Michael F. Barnsley. Since Barnsley has kept his work absolutely secret we still don't know what is possible and what is not. But Fisher's contribution allows us to make a fair guess. Anybody who is interested in the perspectives of image compression through fractals will appreciate this appendix.

The second appendix is written by Carl J. G. Evertsz and Benoit B. Mandelbrot and deals with *multifractal measures,* which is one of the hottest subjects in the current scientific discussion of fractal geometry. Usually we think of fractals as objects having some kind of self-similarity. The discussion of multifractal measures extends this concept to the distributions of quantities (for example, the amount of ground water found at a certain location under the surface). Furthermore, it overcomes some shortcomings of the fractal dimension when used as a tool for measurement in science.

Even with these two important contributions there remain many holes in this book. However, fortunately there are exceptional books already in print that can close these gaps. We list the following only as examples: For portraits of the personalities in the field and the genesis of the subject matter, as well as the scientific background and interrelationships, there are *Chaos — Making a New Science*,[3] by James Gleick, and *Does God Play Dice?*,[4] by Ian Stewart. For the reader who is more interested in a systematic mathematical exposition or who is ready to advance into the depths, there are the following titles: *An Introduction to Chaotic Dynamical Systems*[5] and *Chaos, Fractals, and Dynamics*,[6] both by Robert L. Devaney, and *Fractals Everywhere*,[7] by Michael F. Barnsley. An adequate technical discussion of fractal dimension can be found in the two exceptional texts, *Measure, Topology and Fractal Geometry*,[8] by Gerald A. Edgar, and *Fractal Geometry*,[9] by Kenneth Falconer. Readers more interested in fractals in physics will appreciate *Fractals*,[10] by Jens Feder, while readers who look for fractals in chemistry should not miss *The Fractal Approach to Heterogeneous Chemistry*,[11] by David Avnir. And last but not least, there is the book of books about fractal geometry written by Benoit B. Mandelbrot, *The Fractal Geometry of Nature*.[12]

We owe our gratitude to many who have assisted us during the writing of this book. Our students Torsten Cordes and Lutz Voigt have produced most of the graphics very skillfully and with unlimited patience. They were joined by two more of our students, Ehler Lange and Wayne Tvedt, during part of the preparation time. Douglas Sperry has read our text very carefully at several stages of its evolution and, in addition to helping to get our English de-Germanized, has served in the broader capacity of copy editor. Ernst Gucker, who is working on the German edition, suggested many improvements. Friedrich von Haeseler, Guentcho Skordev, Heinrich Niederhausen and Ulrich Krause have read several chapters and provided valuable suggestions. We also thank Eugen Allgower, Alexander N. Charkovsky,

[3]Viking, 1987.
[4]Penguin Books, 1989.
[5]Second Edition, Addison Wesley, 1989.
[6]Addison Wesley, 1990.
[7]Academic Press, 1989.
[8]Springer-Verlag,1990
[9]John Wiley and Sons, 1990.
[10]Plenum, 1988
[11]Wiley,1989
[12]W. H. Freeman, 1982.

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The entire book has been produced using the TEX and LATEX typesetting systems where all figures (except for the half-tone and color images) were integrated in the computer files. Even though it took countless hours of sometimes painful experimentation setting up the necessary macros it must be acknowledged that this approach immensely helped to streamline the writing, editing and printing.

Finally, we have been very pleased with the excellent cooperation of Springer-Verlag in New York.

Heinz-Otto Peitgen, Hartmut Jürgens, Dietmar Saupe
Bremen, May 1992

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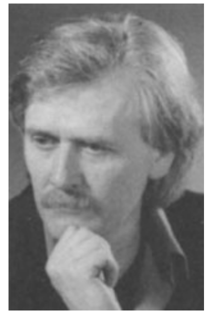

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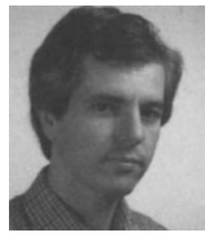

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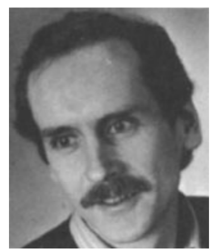

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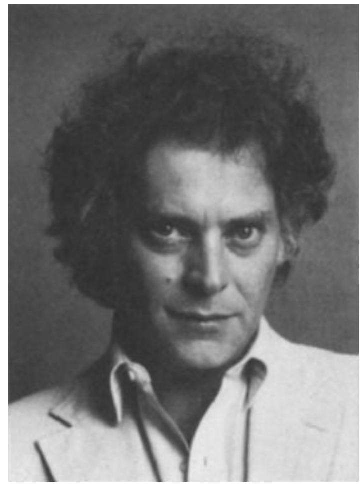
Foreword

Mitchell J. Feigenbaum[1]



Mitchell J. Feigenbaum

The study of chaos is a part of a larger program of study of so-called "strongly" nonlinear systems. Within the context of physics, the exemplar of such a system is a fluid in turbulent motion. If chaos is not exactly the study of fluid turbulence, nevertheless, the image of turbulent, erratic motion serves as a powerful icon to remind a physicist of the sorts of problems he would ultimately like to comprehend. As for all good icons, while a vague impression of what one wants to know is sensibly clear, a precise delineation of many of these quests is not so readily available. In a state of ignorance, the most poignantly insightful questions are not yet ripe for formulation. Of course, this comment remains true despite the fact that for technical exigencies, there are definite questions that one desperately wants the answers to.

Fluid turbulence indeed presents us with highly erratic and only partially predictable phenomena. Historically, since Laplace say, physical scientists have turned to the statistical methods when presented with problems that concern the mutual behaviors of innumerably large numbers of pieces. If for no other reason, one does so to reduce the number of details that one must measure, specify, compute, whatever. Thus, it is easier to say that 43% of the population voted for X than to offer the roster of the behavior of each of millions of voters. Just so, it is easier to specify how many gas molecules there are in an easily measurable volume than to write out the list of where and how fast each one is. This idea is altogether reasonable if not even the most desirable one. However, if one is to work out a theory of

[1]Mitchell J. Feigenbaum, Toyota Professor, The Rockefeller University, New York.

these things, so that a prediction might be rendered, then as in all matters of statistics, one must determine a so-called distribution function. This means a theoretical prediction of just how often out of uncountably many elections, etc. it is expected that each value of this average voter response occurs. For the voter question and the density of a gas question, there is just one number to determine. For the problem of fluid turbulence, even in this statistical quest, one must ask a much richer question: For example, how often do we see eddies of each size rotating at such and such a rate?

The Laws of Fluids

For the problem of voters I don't have any serious idea of how to theoretically determine this requisite distribution; nor with good frequency, do the polls succeed in measuring it. After all, it might not exist in the sense that it rapidly and significantly varies from day to day. However, since physicists have long known quite reliably the laws of fluids — that is, the rules that allow you to deduce what each bit of the fluid will do later if you know what they all do now, there might be a way of doing so. Indeed, the main idea of the branch of physics called statistical mechanics is rooted in the belief that one knows in advance how to do this. The idea is, basically, that each possible detailed configuration occurs with equal likelihood. Indeed the word "chaos" first entered physics in Maxwell's phrase "state of molecular chaos" in the last century to loosely mean this. Statistical mechanics — especially in its quantum mechanical form — works very well indeed, and provides us with some of our most wonderful knowledge. However, altogether regrettably, in the context of fluid turbulence, it has persisted for the last century to roundly fail. It turns out to be a question of truly deducing from the known laws of microscopic motion of fluids what this rule of distribution must be, because the easy guess of "everything is as random as possibly" simply doesn't work. And when that guess doesn't work, there exists as of today no methodology to provide it. Moreover, if in our present state of knowledge we should be forced to appraise the situation, then we would guess that an extraordinarily complicated distribution is required to account for the phenomena: Should it be fractal in nature, then fractal of the most perverse sort. And the worst part is that we really don't possess the mathematical power to generally say what class of object it might be sought among. Remember, we're not looking for a perfectly good quick-fix: If we are serious in seeking understanding of the analytical description of Nature, then we demand much more. When the subject of chaos and a part of that larger program called strongly nonlinear physics shall have been deemed penetrated, we shall know thoroughly how to respond to such questions, and readily image intuitively what the answers look like. To date, we can now compellingly do so for much simpler problems — and have come to possess that capability only within the last decades.

As I have said earlier, I don't necessarily care about turbulence. Rather, it serves as an icon representing a genre of problems. I was trained as a theoretical high energy physicist, and grew deeply troubled that no methods save for that of successive improvements, so-called perturbation methods, existed. Apart from the brilliant effort of Ken Wilson, in his version of the renormalization group, that circumstance is unchanged. Knowing the microscopic laws of how things move — such schemes are called "dynamical systems" — still leaves us almost altogether in the dark as to their larger consequences. Are the theories no good, or is it that we just can't determine what they contain? At the moment it's impossible to say. From high energy physics to fluid physics and astrophysics our inherited ways of thinking mathematically simply fail to serve us. In a way, if perhaps modest, the questions tackled in the effort to comprehend what is now called chaos have faced these questions of methodology head on.

Nonlinearity

Let me now backtrack and discuss nonlinearity. This means first linearity. Linearity means that the rule that determines what a piece of a system is going to do next is not influenced by what it is doing now. More precisely, this is intended in a differential or incremental sense: For a linear spring, the increase of its tension is proportional to the increment whereby it is stretched, with the ratio of these increments exactly independent of how much it has already been stretched. Such a spring can be stretched arbitrarily far, and in particular will never snap or break. Accordingly, no real spring is linear.

The mathematics of linear objects is particularly felicitous. As it happens, linear objects enjoy an identical, simple geometry. The simplicity of this geometry always allows a relatively easy mental image to capture the essence of a problem, with the technicality, growing with the number of parts, basically a detail, until the parts become infinite in number, although often then too, precise answers can be readily determined.

The historical prejudice against nonlinear problems is that no so simple nor universal geometry usually exists. Until recently, the general scientific perception was that a certain nonlinear equation characterized some particular problem. If the specific problem was sufficiently interesting or demanding of resolution, then perhaps, particular methods could be created for it, while however, it was well understood that the travail would probably be of no avail in other contexts.

Perturbation Method

Indeed only one method was well understood and universally learned, the perturbation method. If a linear problem is viewed through distorting lenses, it qualitatively will do the same thing: if it repeated every five seconds it would persist to appear so seen through the lenses. Nevertheless, it would now no longer appear to exhibit equal tension increments for the equal elongations: After all, the tension is measurably unchanged by distorting lenses, whereas all spatial measurements are. That is, the device of distorting lenses turns a linear problem into a nonlinear one. The method of perturbation basically works only for nonlinear problems that are distorted versions of linear ones. And so, this uniquely well-learned method is of no avail in matters that aren't merely distortions of linear ones.

Geometry of Chaos

Chaos is absent in distorted linear problems. Chaos and other such phenomena that are qualitatively absent in linear problems are what we call strongly nonlinear phenomena. It is this failure to subscribe to the spectrum of configurations allowed by distorting a simple geometry that renders these problems anywhere from hard in the extreme to impenetrable. How does one ever start to intelligently describe an awkward new geometry? This question is for example intended to be loosely akin to the question of how one should describe the geometry of the surface of the Earth, not through our abstracted perceptual apparatus that allows us to visualize it immersed within a vastly larger three-dimensional setting, but rather intrinsically, forbidding this use of imagination. The solution of this question, first by Gauss and then extended to arbitrary dimensions by Riemann is, as many of you must know, at the center of the way of thinking of Einstein's General Theory of Relativity, our theory of gravity. What is to be the geometry of the object that describes the turbulent fluid's distribution function? Are there intrinsic geometries that describe various chaotic motions, that serve as a unifying way of viewing these disparate nonlinear problems, as kindred? I ask the question because I know the answer to be affirmative in certain broad circumstances. The moment this is accepted, then strongly nonlinear problems appear no longer as each one its own case, but rather coordinated and suitable for theorizing upon as their own abstract entity. This promotion from the detailed specific to the membership in a significant general class is one of the triumphs of the study of chaos in the last decade or two.

An even stronger notion than this generality of shared qualitative geometry is the notion of universality, which means no less than that this shared geometry is not only one of a qualitative similarity but also one of true quantitative identicality. After what has been, if you will, a long preamble, the fact that strongly nonlinear problems, with surprising frequency, can share a quantitatively identical geometry is what I shall pursue for the rest of this discussion, and constitutes what is termed universality in the transition to chaos.

Universality

In a qualitative way of thinking, universality can be seen to be not so surprising. There are two arguments to support this. The first part has simply to do with nonlinearity. Just as a linear object has a constant coefficient of proportionality between, for example, its tension and its expansion, a similar, but nonlinear version, has an *effective* coefficient *dependent* upon its extension. So, consider two completely different nonlinear systems. By adjusting things correctly it is not inconceivable that the effective coefficients of each part of each of the two systems could be set the same so that then their behaviors could, at least initially, be identical. That is, by setting some numerical constants (properties, so to speak, that specify the environment, mathematically called 'parameters') *and* the actual behaviors of these two systems, it is possible that they can do the identical thing. For a linear problem this is ostensibly true: For systems with the same number of parts and mutual connections, a freedom to adjust all the parameters allows one to be adjusted to be identical (truly) to the other. But, for many pieces, this is many adjustments. For a nonlinear system, adjusting a small number of parameters can be compensated, in this quest for identical behavior, by an adjustment of the momentary positions of its pieces. But then it must be that not all motions can be so duplicated between systems.

Thus, the first part of the argument is that nonlinearity confers a certain flexibility upon the adaptability of an object to desirable behavior. Nevertheless, should the precise adjustment of too many specific and subtle details be required in order to achieve a certain universal behavior, then the idea would be pedantic at best.

The Monadology of Leibniz

However, there is a second more potent argument, a paraphrasing of Leibniz in "The Monadology" which can render this first argument potent. Let us contemplate that the motion we intend to determine to be universal over nonlinear systems has arisen by the successive imposition of more and more qualitative constraints. Should this growingly large host of impositions prove to be generally amenable to such systems (this is the hard and a priori neither obvious nor reasonable part of the discussion) then we shall ultimately discover these disparate systems to all be identically constrained by an infinite number of qualitative and if you will, self consistent, requirements. Now, following Leibniz, we ask, "In how many precise, or *quantitative*, ways can this situation be tenable?" And we respond, following Leibniz, by asserting in precisely one possible uniquely determined way.

This is the best verbalization I know how to offer to explain why such a universal behavior is possible. Both mathematics and physical experimentation confirm its rectitude perfectly. But it is perhaps difficult to have you realize how extraordinary this result appeared given the backdrop of physical and mathematical thinking in 1976 when it first appeared together with its full conceptual analysis. As anecdotal evidence, I had been directed to expound these results to one of the great mathematicians who is renowned for his results on dynamical systems. I spoke with him at the very end of 1976. I kept trying to tell him that there was a complete *quantitative* universality to these phenomena, and he equally often understood me to have duplicated some known *qualitative* results. Finally he said "You mean to tell me these are metrical results?" (Metrical is a mathematical code word that means quantitative.) And I said "Yes." "Well, then you're wrong!" he asserted, and turned his back on me to terminate the conversation.

The Scientific Method

Anecdote aside, what is remarkable about all this? First of all, an easy piece of methodological insight. As practitioners of a truly analytical science, physicists were trained to know that qualitative explanations are insufficient to base truth upon. Quite to the contrary, it is regarded to be at the heart of the "scientific method" that ever more precise measurements will discriminate between rival quantitative theories to ultimately select out one as the correct encoding of the qualitative content. (Thus, think of geocentric versus heliocentric planetary theories, both qualitatively explaining the retrograde motions of the planets.) Here the method is turned on its head: Qualitatively similar phenomena, independent of any other ideational input must ineluctably lead to the measurably *identical* quantitative result. Whence the total phenomenological support for this mighty "scientific method?"

How Universality Works

Secondly, a new principle of "economy" immediately emerges. Why put out Herculean efforts to calculate the consequences of some particular and highly difficult encoding of physical laws, when anything else, however trivial, possessing the same qualitative properties will yield *exactly* the same predictions and results? And this all the more satisfying, since one doesn't even *know* the exact equations that describe various of these phenomena, fluid phenomena in particular. Because, these phenomena have nothing to do, whatsoever, with the detailed, particular, microscopic laws that happen to be at play. This aspect, that is, of substituting easy problems for hard ones with no penalty has been, as a way of thinking and performing research, the prominent fruit of the recognition of universality. When can it work? Well, in complicated interactions of scores of chemical species, in laser phenomena, in solid state phenomena, in, at least only partially, biological rhythmic phenomena such as apneas and arhythmias, in fluids and of course, in mathematics.

The Essence of Chaos

But now, as I move towards the end of this claim for virtue, let me discuss "chaos" a bit more per se and revisit my opening "preamble." Much of chaos as a science is connected with the notion of "sensitive dependence on initial conditions." Technically, scientists term as "chaotic" those nonrandom complicated motions that exhibit a very rapid growth of errors that, despite perfect determinism, inhibits any pragmatic ability to render accurate long-term prediction. While nomenclaturally speaking, this is perforce true, I personally am not most intrigued nor concerned with this facet of my subject. I've never told you what the "transition to chaos" means, but you can readily guess from the verbiage that it's something that starts off not being chaotic, ends up being so, and hence somehow passes from one to the next. The most important fact is that there is a discernibly precise "moment", with a corresponding behavior, which is neither chaotic nor nonchaotic, at which this transition occurs. Yes, errors do grow, but only in a marginally predictable, rather than in an unpredictable fashion. In this state of marginal predictability inheres embryonically all the seeds of the chaotic behavior to come. That is, this transitional point, the legitimate child of universality, without full-fledged sensitive dependence upon initial conditions knows fully how to dictate to its progeny in turn how this latter phenomenon must unfold. For a certain range of possible behaviors of strongly nonlinear systems, this range surrounding the transition to chaos, the information obtained just at the transition point fully organizes the spectrum of behaviors that these chaotic systems can exhibit.

The Geometry of Chaos

Now what is it that turns out to be universal? The answer, mostly, is a precise quantitative determination of the intrinsic geometry of the space upon which this marginal chaotic motion lives together with the full knowledge of how in the course of time this space is explored. Indeed, it was from the analysis of universality at the transition to chaos that we have come to recognize the precise mathematical object that fully furnishes the intrinsic geometry of these sort of spaces. This object, a so-called scaling function, together with the mathematically precise delineation of universality, constitutes one of the major results of the study of chaos. Granted the broad range of objects that can be termed fractal, these geometries are fractal. But not the heuristic sort of 'dragons', 'carpets', 'snowflakes', etc. Rather, these are structures which are elaborated upon at smaller and smaller scales differently at each point of the object, and so are infinitely more complicated than the above heuristic objects. There is, in more than just a way of speaking, a geometry of these dynamically created objects, and that geometry requires a scaling function to fully elucidate it. Many of you are aware of the existence of a certain object called the 'Mandelbrot set'. Virtually none of you, though, even having simulated it on your own computers, are aware that its ubiquitous existence in those sufficiently smooth contexts in which it appears, is the consequence of universality at the transition of chaos. Every one of its details is implicit in those embryonic seeds I have mentioned before.

Thus, the most elementary consequence of this deep universal geometry is that, in gross organization we notice a set of discs — the largest the main cardioid — one abutting upon the next and of rapidly diminishing radii. How rapidly do they diminish in size? In fact, each one is δ times smaller that its predecessor, with δ, a universal constant, approximately equal to $4.6692016\ldots$, the best known of the constants that characterize universality at the transition of chaos.

I have now come around full circle to my introductory comments. We have, in the last decade, succeeded in coming to know many of the correct ideas and their mathematical language in regard to the question, 'What is the nature of the objects upon which we see our statistical distributions?' 'Dimension' is a mathematical word possessing a quite broad range of technical connotations. Thus, the theory of universality is erected in a very low (that is, one- or two-) dimensional setting. However the information discussed is of an infinite-dimensional character. The physical phenomena exhibiting these behaviors can appear, for example, in the physical three-dimensional space of human experience, with the number of interacting, cooperating pieces that comprise the system investigated — also a statement of its dimension — either merely a few or an infinitude. Nevertheless, our understanding to date is of what must be admitted to be a relatively simple set of phenomena — relatively simple in comparison to the swirling and shattering complexity of fluid motions at the foot of a waterfall, phenomena that loom large and deeply impress upon us how much lies undiscovered before us.