TEXTS AND READINGS IN MATHEMATICS **33**

# Introduction to Probability and Measure

# Contents

**Preface** xi

**Preface to the Revised Edition** xiii

**Acknowledgements** xv

**1 Probability on Boolean Algebras** 1
- 1.1 Sets and Events . . . 1
- 1.2 Probability on a Boolean algebra . . . 4
- 1.3 Probability Distributions and Elementary Random Variables . . . 8
- 1.4 Repeated Trials and Statistical Independence . . . 20
- 1.5 The Poisson Approximation to the Binomial Distribution 28
- 1.6 The Normal Approximation to the Binomial Distribution 30
- 1.7 The Multivariate Normal Approximation to the Multinomial Distribution . . . 33
- 1.8 Some Applications of the Normal Approximation . . . 35
- 1.9 Independent Simple Random Variables and Central Limit Theorem . . . 39
- 1.10 Conditional Probability . . . 43
- 1.11 Laws of Large Numbers . . . 48
- 1.12 An Application of the Law of Large Numbers to a Problem in Analysis . . . 53

**2 Extension of Measures** 56
- 2.1 $\sigma$- algebras and Borel Spaces . . . 56
- 2.2 Monotone Classes . . . 59
- 2.3 Measures on Boolean Semialgebras and Algebras . . . 61
- 2.4 Extension of Measures to $\sigma$-Algebras . . . 69
- 2.5 Uniqueness of Extension of Measures . . . 73

2.6 Extension and Completion of Measures . . . . . 73
2.7 Measures on Metric Spaces . . . . . 77
2.8 Probability Contents . . . . . 84
2.9 The Lebesgue Measure on the Real Line . . . . . 92

**3 Borel Maps** **96**
3.1 Elementary Properties of Borel Maps . . . . . 96
3.2 Borel Maps into Metric Spaces . . . . . 99
3.3 Borel Maps on Measure Spaces . . . . . 103
3.4 Construction of Lebesgue Measure and Other Measures in the Unit Interval through Binary, Decimal and other $k$-ary Expansions . . . . . 113
3.5 Isomorphism of Measure Spaces . . . . . 118
3.6 Measures on Projective Limits of Borel Spaces . . . . . 122

**4 Integration** **132**
4.1 Integration of Nonnegative Functions . . . . . 132
4.2 Integration of Borel Functions . . . . . 138
4.3 Integration of Complex Valued Functions . . . . . 144
4.4 Integration with Respect to a Probability Measure . . . . 145
4.5 Riemann and Lebesgue Integrals . . . . . 146
4.6 Riesz Representation Theorem . . . . . 148
4.7 Some Integral Inequalities . . . . . 160

**5 Measures on Product Spaces** **173**
5.1 Transition Measures and Fubini's Theorem . . . . . 173
5.2 Convolution of Probability Measures on $R^n$ . . . . . 182
5.3 The Lebesgue Measure on $R^n$ . . . . . 185
5.4 The Convolution Algebra $L_1(R^n)$ . . . . . 196
5.5 Approximation in $L_p$ Spaces with respect to Lebesgue Measure in $R^n$. . . . . 197

**6 Hilbert Space and conditional Expectation** **204**
6.1 Elementary Properties of Banach Spaces . . . . . 204
6.2 Projections in a Hilbert Space . . . . . 208
6.3 Orthonormal Sequences . . . . . 219
6.4 Completeness of Orthogonal Polynomials . . . . . 226
6.5 Conditional Expectation . . . . . 234
6.6 Conditional Probability . . . . . 247
6.7 Regular Conditional Probability Distributions . . . . . 249

6.8 Radon-Nikodym Theorem and Lebesgue Decomposition . 257
6.9 Elementary Properties of Radon-Nikodym Derivatives . . 261
6.10 Law of Large Numbers and Ergodic Theorem . . . . . 266
6.11 Dominated Ergodic Theorem . . . . . 278

**7 Weak Convergence of Probability Measures 281**
7.1 Criteria for Weak Convergence in the Space of Probability Measures . . . . . 281
7.2 Prohorov's Theorem . . . . . 288
7.3 Fourier Transforms of Probability Measures in $R^k$ . . . . . 294

**8 Invariant Measures on Groups 309**
8.1 Haar Measure . . . . . 309
8.2 Quasi Invariant Measures on Homogeneous Spaces . . . . 316
8.3 Mackey-Weil Theorem . . . . . 324