Graduate Texts in Mathematics 2

John C. Oxtoby

Measure and Category

A Survey of the Analogies between Topological and Measure Spaces

Second Edition

Springer-Verlag
New York Heidelberg Berlin

John C. Oxtoby
Department of Mathematics
Bryn Mawr College
Bryn Mawr, PA 19010
USA

AMS Subject Classification (1980):
26 A 21, 28 A 05, 54 C 50, 54 E 50, 54 H 05, 26-01, 28-01, 54-01

Library of Congress Cataloging in Publication Data

Oxtoby, John C.
Measure and category.

(Graduate texts in mathematics; 2)
Bibliography: p.
Includes index.
1. Measure theory. 2. Topological spaces. 3. Categories (Mathematics)
I. Title. II. Series. QA312.09 1980 515.4'2 80-15770

Softcover reprint of the hardcover 2nd edition 1971

9 8 7 6 5 4 3 2 1

ISBN 978-1-4684-9341-2 ISBN 978-1-4684-9339-9 (eBook)
DOI 10.1007/978-1-4684-9339-9

Preface to the Second Edition

In this edition, a set of Supplementary Notes and Remarks has been added at the end, grouped according to chapter. Some of these call attention to subsequent developments, others add further explanation or additional remarks. Most of the remarks are accompanied by a briefly indicated proof, which is sometimes different from the one given in the reference cited. The list of references has been expanded to include many recent contributions, but it is still not intended to be exhaustive.

Bryn Mawr, April 1980 John C. Oxtoby

Preface to the First Edition

This book has two main themes: the Baire category theorem as a method for proving existence, and the "duality" between measure and category. The category method is illustrated by a variety of typical applications, and the analogy between measure and category is explored in all of its ramifications. To this end, the elements of metric topology are reviewed and the principal properties of Lebesgue measure are derived. It turns out that Lebesgue integration is not essential for present purposes—the Riemann integral is sufficient. Concepts of general measure theory and topology are introduced, but not just for the sake of generality. Needless to say, the term "category" refers always to Baire category; it has nothing to do with the term as it is used in homological algebra.

A knowledge of calculus is presupposed, and some familiarity with the algebra of sets. The questions discussed are ones that lend themselves naturally to set-theoretical formulation. The book is intended as an introduction to this kind of analysis. It could be used to supplement a standard course in real analysis, as the basis for a seminar, or for independent study. It is primarily expository, but a few refinements of known results are included, notably Theorem 15.6 and Proposition 20.4. The references are not intended to be complete. Frequently a secondary source is cited where additional references may be found.

The book is a revised and expanded version of notes originally prepared for a course of lectures given at Haverford College during the spring of 1957 under the auspices of the William Pyle Philips Fund. These, in turn, were based on the Earle Raymond Hedrick Lectures presented at the Summer Meeting of the Mathematical Association of America at Seattle, Washington, in August, 1956.

Bryn Mawr, April 1971 John C. Oxtoby

Contents

1. Measure and Category on the Line 1
Countable sets, sets of first category, nullsets, the theorems of Cantor, Baire, and Borel

2. Liouville Numbers 6
Algebraic and transcendental numbers, measure and category of the set of Liouville numbers

3. Lebesgue Measure in r-Space 10
Definitions and principal properties, measurable sets, the Lebesgue density theorem

4. The Property of Baire 19
Its analogy to measurability, properties of regular open sets

5. Non-Measurable Sets 22
Vitali sets, Bernstein sets, Ulam's theorem, inaccessible cardinals, the continuum hypothesis

6. The Banach-Mazur Game 27
Winning strategies, category and local category, indeterminate games

7. Functions of First Class 31
Oscillation, the limit of a sequence of continuous functions, Riemann integrability

8. The Theorems of Lusin and Egoroff 36
Continuity of measurable functions and of functions having the property of Baire, uniform convergence on subsets

9. Metric and Topological Spaces 39
Definitions, complete and topologically complete spaces, the Baire category theorem

10. Examples of Metric Spaces 42
Uniform and integral metrics in the space of continuous functions, integrable functions, pseudo-metric spaces, the space of measurable sets

11. Nowhere Differentiable Functions 45
Banach's application of the category method

12. The Theorem of Alexandroff 47
Remetrization of a G_δ subset, topologically complete subspaces

13. Transforming Linear Sets into Nullsets 49
The space of automorphisms of an interval, effect of monotone substitution on Riemann integrability, nullsets equivalent to sets of first category

14. Fubini's Theorem 52
Measurability and measure of sections of plane measurable sets

15. The Kuratowski-Ulam Theorem 56
Sections of plane sets having the property of Baire, product sets, reducibility to Fubini's theorem by means of a product transformation

16. The Banach Category Theorem. 62
Open sets of first category or measure zero, Montgomery's lemma, the theorems of Marczewski and Sikorski, cardinals of measure zero, decomposition into a nullset and a set of first category

17. The Poincaré Recurrence Theorem 65
Measure and category of the set of points recurrent under a nondissipative transformation, application to dynamical systems

18. Transitive Transformations 70
Existence of transitive automorphisms of the square, the category method

19. The Sierpinski-Erdös Duality Theorem 74
Similarities between the classes of sets of measure zero and of first category, the principle of duality

20. Examples of Duality 78
Properties of Lusin sets and their duals, sets almost invariant under transformations that preserve nullsets or category

21. The Extended Principle of Duality 82
A counter example, product measures and product spaces, the zero-one law and its category analogue

22. Category Measure Spaces 86
Spaces in which measure and category agree, topologies generated by lower densities, the Lebesgue density topology

Supplementary Notes and Remarks 93

References 101

Supplementary References 102

Index 105