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# Complexity of Proofs and Their Transformations in Axiomatic Theories

V. P. Orevkov

American Mathematical Society

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# СЛОЖНОСТЬ ДОКАЗАТЕЛЬСТВ И ИХ ПРЕОБРАЗОВАНИЙ В АКСИОМАТИЗИРОВАННЫХ ТЕОРИЯХ

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# Contents


Introduction 1

CHAPTER I. Upper Bounds on Deduction Elongation in Cut Elimination 9
§1. The calculi $KGL(\mathfrak{A})$ and $IGL(\mathfrak{A})$ 9
§2. Measures of the complexity of proofs 13
§3. Admissibility of structural rules 17
§4. Cut elimination in $KGL(\mathfrak{A})$ and $IGL(\mathfrak{A})$ 20
§5. The calculi $KH(\mathfrak{A})$ and $IH(\mathfrak{A})$ 25

CHAPTER II. Systems of Term Equations with Substitutions 31
§6. Systems of term equations with substitutions. Main lemmas 31
§7. Extension tree of a CTS-system 40
§8. Representation of enumerable sets by TS-systems 51
§9. Upper bounds on the height of natural solutions of systems of linear Diophantine equations 56
§10. Upper bound on the periodicity index of solutions of CTS-systems 63
§11. An algorithm deciding the existence of solutions of restricted substitution width 76

CHAPTER III. Logical Deduction Schemata in Axiomatized Theories 81
§12. Systems of equations in formulas 81
§13. Deduction schemata in axiomatized Hilbert-type theories 94
§14. Deducibility of a formula in accordance with a given schema 99
§15. Deduction schemata in Gentzen calculi 106
§16. Rebuilding of proofs on the level of schemata 114

CHAPTER IV. Bounds for the Complexity of Terms Occurring in Proofs 119
§17. Comparison of the lengths of direct and indirect proofs of existence theorems in the predicate calculus 119
§18. Complexity version of the existence property of the constructive predicate calculus 126

CHAPTER V. Proof Strengthening Theorems 131
§19. Proof strengthening theorems in finitely axiomatized theories 131
§20. Proof strengthening theorems in formal arithmetic 134
§21. Upper and lower bounds on lengths of deductions in formal arithmetics 144

References 151