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Lie Groups and Lie Algebras III

Structure of Lie Groups
and Lie Algebras



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Structure of Lie Groups and Lie Algebras

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