

# ORDINARY DIFFERENTIAL EQUATIONS

*An Elementary Textbook for  
Students of Mathematics, Engineering,  
and the Sciences*

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