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Numerical Analysis of Compressible Fluid Flows

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The cover picture depicts vortex structures arising in the Kelvin-Helmholtz problem. Computation was done by the second order finite volume method based on the generalized Riemann problem.

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