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# Nonlinear Functional Analysis and its Applications

II/A: Linear Monotone Operators

Translated by the Author and by Leo F. Boron†

With 45 Illustrations



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