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# Foliations on Surfaces

With 23 Figures

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Моим дорогим, маме и папе

If you can dream – and not make dreams your master,
If you can think – and not make thoughts your aim..

Rudyard Kipling : 'If–'
*("Poems. Short Stories")*

We are what we think.
All that we are arises with our thoughts,
With our thoughts we make the World...

Dhammapada : Choices
*("The Sayings of the Buddha")*

# Foreword

H. Poincaré, the founder of the qualitative theory of differential equations, was the first to realize the significance of the simple fact that the trajectories of a smooth vector field, $v$, determine a geometric picture called the *phase portrait* of $v$. From a general viewpoint, such a picture can be considered as a *locally oriented one-dimensional foliation, $\mathcal{F}$, with singularities.* Actually, these two geometric concepts are equivalent because for a given smooth foliation $\mathcal{F}$ one can determine the corresponding vector field $v$ up to a change of time.

There is a vast literature devoted to the case of locally oriented foliations on two-dimensional manifolds. Let me only mention two recent texts: S. Aranson, G. Belitsky, E. Zhuzhoma, *Introduction to qualitative theory of dynamical systems on surfaces*, AMS Math. Monographs, 1996; I. Nikolaev, E. Zhuzhoma, *Flows on two-dimensional manifolds*, Lecture Notes in Math., 1705, Springer Verlag, 1999. But there exist foliations on surfaces that are non-orientable at some singular points. Indeed, consider a saddle point with an *odd* number, $n$, of separatrices. If $n = 1$ ($n = 3$), the singularity is called a *thorn* (*tripod*, respectively). A reader with a vivid imagination can recognize thorns and tripods even in her (his) own fingerprints. Of course, there are really interesting fields where thorns and tripods are met (for example, the theory of liquid crystals). Although the above mentioned books are devoted mainly to the locally oriented case, a few thorns and tripods flashed there on the horizon.

The present book gives an account of basic results obtained in exploration of foliations on surfaces with emphasis on the *locally non-orientable* case. It demonstrates that the general setting of not necessarily locally orientable foliations leads to a better understanding of some classical problems initially investigated by A. Cayley, G. Darboux and É. Picard. Various approaches to the theory of foliations are discussed and some unsolved problems formulated.

I invite the reader interested in geometry and analysis on surfaces to visit the refreshingly new land of locally non-orientable foliations and enjoy the world of thorns, tripods, apples, sun-sets, labyrinths and many other exotic things (including a foliation with a leaf everywhere dense in a disk).

*Idel Bronshteyn*

# Preface

A foliated space is one of the fundamental concepts of modern geometry and topology. Given a topological space $X$, by a foliation, $\mathcal{F}$, one understands a partition of $X$ into a disjoint sum of 'leaves' with a regular 'microscopic' behaviour.

For example, if $X$ is an $n$-dimensional manifold, then every small part of $\mathcal{F}$ looks like a family of the parallel planes $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^{n-k}$. The global behaviour of leaves can be quite tricky and one of the goals of the theory is to understand the asymptotic properties of leaves.

The theory of foliations begins with the work of Ch. Ehresmann and G. Reeb in the 1940's. Perhaps they were inspired by the ideas of H. Poincaré, A. M. Lyapunov and G. D. Birkhoff who promoted geometric methods in the study of differential equations.

The development of the subject since the 1940s is reflected in a beautiful survey by B. Lawson, *Foliations*, Bull. Amer. Math. Soc. 80 (1974), 369-418. The advances in the area are due to D. V. Anosov, R. Bott, L. Conlon, A. Connes, A. Haefliger, G. Hector, R. Hermann, F. W. Kamber J. W. Milnor, S. P. Novikov, B. Reinhart, H. Rosenberg, P. Schweitzer, H. Seifert, I. Tamura, W. P. Thurston and P. Tondeur, to name a very few. The younger generation is represented by G. Gabai, E. Ghys, C. Godbillon, S. Hurder, J. Martinet, S. Matsumoto, P. Molino, R. Moussu, J. F. Plante, R. Sacksteder, D. Tischler, T. Tsuboi, P. Walczak and J. Vey; see the vast bibliography of the book by C. Godbillon, *Feuilletages. Étude géometriques*, Birkhäuser, Basel-Boston-Berlin, 1991.

A modest aim of the present monograph is to cover the case when $X$ is a two-dimensional manifold (a surface).

Let $k$ be a dimension of leaves. In the case $k = 0$ (0-dimensional leaves) the surface is foliated by points, which is trivial. The case $k = 1$ assumes that the Euler characteristic of $X$ vanishes. In other words, only the torus and Klein bottle can carry such a foliation. This theory received a full treatment in the monograph of G. Hector and U. Hirsch, *Introduction to the Geometry of Foliations*, Parts A and B, Vieweg. 1981.

The point of view adopted in this book is that foliations can be *singular.* In other words, the dimension of leaves of $\mathcal{F}$ is not fixed and it can vary from point to point, being either 1 or 0. Moreover, until otherwise mentioned, the set Sing $\mathcal{F}$ is assumed to be finite. In this setting, by a (singular) foliation $\mathcal{F}$ on a surface $X$ one means a partition of $X\setminus$ Sing $\mathcal{F}$ into disjoint curves (1-leaves), this partition being locally homeomorphic to a family of parallel lines. As far as the singularity set Sing $\mathcal{F}$ (0-leaves) is concerned, it must be defined separately for each class of foliation.

An immediate example of singular foliations is given by a flow $v^t$ on $X$. The reader can see that 1-leaves correspond to the trajectories of $v^t$ and singular points to the fixed points of $v^t$. This simplicity is false, since the majority of foliations can be given by no flow whatsoever.

There are two kinds of obstacles to turning a foliation into a flow. The local obstacle is represented by a *non-orientable singularity*, i.e. a singularity whose 'phase portrait' cannot be coherently oriented along the leaves. Many examples are known (thorns, tripods, etc).

The global obstacle is given by a *labyrinth*, i.e. a collection of leaves which are everywhere dense in the disk. The reader can take a look at Fig. 0.3 to see the idea. The obstacle is the Poincaré-Bendixson theorem which says that the long-time behaviour of trajectories in the disk is either periodic or stationary.

There are however parallels between the two theories. To encourage the reader at this point, we shall mention that eventually every foliation can be represented by a ($\mathbb{Z}_2$-symmetric) flow on a surface with involution. This procedure is called a *normalization* and will be discussed later on.

The theory of surface foliations is as old as the number theory. The problem of approximation of an irrational number using rationals is linked to the geometry of foliations. This fact was recognized by Leopold Kronecker in the middle of the 19th century. The irrational (Kronecker's) foliations on the torus have been applied to the problems of celestial mechanics by H. Poincaré, A. Denjoy and T. Cherry. The study of the geometry and topology of foliations on surfaces was undertaken by S. Kh. Aranson, T. O'Brien, C. Gardiner, V. Z. Grines, C. Gutierrez, T. Inaba, R. Langevin, G. Levitt, N. Markley, D. Neumann, Ch. Pugh, E. V. Zhuzhoma and others. The theory of measure preserving transformations linked to a surface foliation is due to V. I. Arnold, G. Forni, M. Herman, J. Hubbard, A. Katok, M. Keane, M. Kontsevich, M. Martens, H. Masur, W. de Melo, P. Mendes, E. A. Sataev, Ya. Sinai, S. van Strien, W. A. Veech, J. C. Yoccoz and A. Zorich.

**Cayley-Darboux-Picard Problem and Normalization.** The doctoral dissertation of H. Poincaré, *Sur les courbes défenie par les équations différentielles*, C. R. Acad. Sci. Paris 90 (1880), 673-675, was devoted to the geometric theory of singular points of differential equations. He classified all simple singular points of planar (analytic) vector fields by showing that there are only three types of such points: a simple saddle, a node and a focus (the last two types are topologically equivalent).

Around 1895 É. Picard (*Sur les points singuliers des équations différentielles du premier ordre*, Math. Annalen 46, 521-528) was looking for a similar classification of the non-orientable singular points. He was probably influenced by an earlier work of A. Cayley, *On differential equations and umbilici*, Philos. Mag. 26 (1863), 373-379, 441-452, Coll. Works: V. 6.

In 1896 G. Darboux (*Sur la forme des lignes de courbure dans la voisinage d'un ombilic*, Lecons sur la Theorie des Surfaces, IV, Note 7, Gauthier Villars, Paris) used a geometric approach to settle the same problem.

Unfortunately, from our point of view, no satisfactory theory was built at that time. Let us call the following question, "Find a regular procedure of resolving the non-orientable singularities of planar foliations" the *Cayley–Darboux–Picard problem.*

Several research teams in Brazil (V. Guinez, C. Gutierrez, J. Sotomayor), France (J. F. Mattei, M. F. Michel), Russia [1] (A. A. Kadyrov, A. G. Kuzmin) and USA (B. Smyth, F. Xavier) were involved in this problem for different reasons.

In a short note, *Singular points of the line element fields in the plane*, Izv. Akad. Nauk Resp. Moldova, Matematika 3(9), 1992, 23-29, I. U. Bronstein (Bronshteyn) and the author suggested a method leading to a solution of the Cayley-Darboux-Picard problem. The idea was to consider a double covering to the phase space of the differential equation studied by Cayley, Darboux and Picard. A surprising consequence was a short list of simple (structurally stable) singular points. These appeared to be thorns, tripods, sun-sets and apples.

We call the method a *normalization* (in the sense of Riemann) since it reduces planar foliations to the planar vector fields with a symmetry. This book gives a systematic account of the normalization method.

**Final Remarks.** This book is addressed to graduate students. It will be useful to the specialists in geometry and topology, ergodic theory, dynamical systems, complex analysis, differential and noncommutative geometry.

*Toronto, September 2000* *Igor Nikolaev*

[1] The author learned about this problem from A. A. Kadyrov while working as a postdoctoral researcher in St. Petersburg.

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# Contents

## Part III. Appendix

# Index of Notation

| | |
|---|---|
| $\phi^t$ | continuous flow |
| $O(x)$ | orbit through the point $x$ |
| $O^{\pm}(x)$ | forward (backward) orbit through the point $x$ |
| $\alpha(x)$ | $\alpha$–limit set of the point $x$ |
| $\omega(x)$ | $\omega$–limit set of the point $x$ |
| $\mathcal{Q}(x)$ | quasiminimal set of the orbit $O(x)$ |
| $\Lambda$ | minimal set of the flow $\phi^t$ |
| $(M, \pi^t, \theta)$ | foliation $\mathcal{F}$ given by the flow $\pi^t$ and the involution $\theta$ on the surface $M$ |
| $\theta$ | involution defined by the foliation $\mathcal{F}$ |
| $Sing\ \mathcal{F}$ | set of 0–dimensional leaves of the foliation $\mathcal{F}$ |
| $\mathcal{F}$ | foliation |
| $TM(T^*M)$ | tangent (cotangent) bundle over the manifold $M$ |
| $\omega$ | differential 1–form on $M$ |
| $L_v$ | Lie derivative along the vector field $v$ |
| $i_v$ | inner product in the space of differential forms |
| $BM$ | bivector bundle over the manifold $M$ |
| $Susp\ d$ | suspension over the interval exchange transformation $d$ |
| $\mathcal{L}$ | labyrinth on $M$ |
| $\mathcal{F}_1 \# \mathcal{F}_2$ | sum of the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ |
| $C^r$ | differentiability class of the foliation |
| $\mathcal{F}^r(M)$ | space of $C^r$–smooth foliations on the manifold $M$ |
| $\mathcal{F}^r_0(M)$ | space of structurally stable (rough) foliations on the manifold $M$ |
| $X^R$ | graph $X$ endowed with the rotation system $R$ |
| $\|\mathcal{Q}\|$ | number of the disjoint quasiminimal sets of the foliation $\mathcal{F}$ |
| $g$ | genus of the orientable surface $M$ |
| $p$ | genus of the non orientable surface $N$ |
| $\mathbb{H}$ | Lobachevsky half–plane endowed with the hyperbolic metric |
| $SL_2(\mathbb{Z})$ | modular group |
| $G$ | finite group |
| $Cay\ G$ | Cayley graph of the group $G$ |
| $H_1(M, \mathbb{R})$ | first real homology group of the surface $M$ |
| $H^1(M, \mathbb{R})$ | first real cohomology group of the surface $M$ |
| $\pi_1(M)$ | fundamental group of the surface $M$ |

| | |
|---|---|
| $l$ | curve on the surface $M$ |
| $\widetilde{l}$ | preimage of the curve $l$ on the universal covering to $M$ |
| $\partial\mathbb{H}$ | absolute |
| $\mu$ | invariant measure of the flow $\phi^t$ |
| $AF$ | approximately finite $C^*$–algebra $A$ |
| $A_\alpha$ | irrational rotation algebra |
| $\Gamma(n)$ | principal congruence subgroup of the modular group |
| $A(\phi)$ | Artin number of the minimal flow $\phi^t$ |
| $K_0(A)$ | Grothendieck group of the $C^*$–algebra $A$ |
| $\phi(z)dz^2$ | quadratic differential on the surface $M$ |
| $Arg\ \phi(z)$ | argument of the quadratic differential $\phi(z)dz^2$ |
| $\Omega^1(\Omega^2)$ | first (the second) fundamental forms of the surface $M$ |
| $\mathbb{C}^*$ | Riemann sphere $\mathbb{C} \cup \{\infty\}$ |
| $PSL(2,\mathbb{C})$ | group of Möbius transformations of the complex plane $\mathbb{C}$ |
| $T_g$ | Teichmüller space of the Riemann surface of genus $g$ |
| $\mathbb{P}_2$ | complex projective plane |
| $C$ | complex projective curve on $\mathbb{P}_2$ |
| $\wp(z)$ | Weierstrass function |
| $D$ | divisor of the non singular projective curve $C$ |