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Treatise on the Shift Operator

Spectral Function Theory

With an Appendix
by S. V. Hruščev and V. V. Peller

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by Jaak Peetre

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Preface

This book is an elementary introduction to non-classical spectral theory. After the basic definitions and a reduction to the study of the functional model the discussion will be centered around the simplest variant of such a model which, formally speaking, comprises only the class of contraction operators with a one-dimensional rank of non-unitarity ($\operatorname{rank}(I - T^*T) = \operatorname{rank}(I - TT^*) = 1$). The main emphasis is on the technical side of the subject, the book being mostly devoted to a development of the analytical machinery of spectral theory rather than to this discipline itself. The functional model of Sz.-Nagy and Foiaş reduces the study of general operators to an investigation of the compression

$$T = PS|K$$

of the shift operator S, $Sf = zf$, onto coinvariant subspaces (i.e. subspaces invariant with respect to the adjoint shift S^*). In the main body of the book (the "Lectures" in the proper meaning of the word) this operator acts on the Hardy space H^2 and is itself a part of the operator of multiplication by the independent variable in the space L^2 (in the case at hand L^2 means $L^2(\mathbb{T})$, $\mathbb{T}$ being the unit circle), this operator again being fundamental for classical spectral theory. The language of functional models is the language of function theory – or more specifically – Nevanlinna multiplicative function theory, which allows one to apply in connection with problems of spectral theory, not only the rather limited set of tools which were used in the classical approach (the theorems of Liouville and Phragmén-Lindelöf, various compactness principles) but literally the entire theory of boundary behavior of analytic functions as developed in the past 60 years. In the last few years an opposite trend has also made itself noticeable. With the aid of the concept of functional model there are formulated and solved problems which traditionally belonged to the theory of functions of one complex variable. By way of illustration we can mention the theorem on interpolation with unbounded multiplicities, as set forth in Lecture IX, and many further classical results on boundary behavior, which in these Lectures are derived by purely operator-theoretic considerations (uniqueness theorems, the theorem of the Riesz brothers on analytical measures etc.). Quite generally one may say that the clarification of the connections between the fundamental problems of function theory and spectral theory is also the main subject of these Lectures.

The bulk of the "Lectures" (Lecture IV, VI–X) is devoted to a clarification of the rôle of the celebrated corona theorem by Carleson. Around this central theme, in spectral language amounting to a study of unconditionally convergent spectral expansions, we have grouped some other themes – interpolation in various classes of analytic functions, the description of invariant subspaces in terms of multiplicative representations of the "characteristic function" of the operator, approximation theoretic consequences etc. In particular these last consequences will be paid special attention; we shall notably study the question of the completeness of various systems of functions connected with the shift operator: completeness of polynomials with respect to weights, remainders of the Taylor series ($S^{*n}f$, $n \geq 0$), powers of suitable convolution operators, systems of rational functions. These approximation problems are connected with one of three digressions from the "scalar" variant of the functional model referred to above. We consider here problems on the multiplicity of the spectrum and on generating sets for the multiple shift (the operator S_E in the space of vector valued H^2-functions), which are essential both for the theory itself as well as for its ramifications (for instance, in the theory of stationary stochastic processes). The other two digressions concern the theory of "the class C_0", defined by the condition that for its members an analogue of the Hamilton-Cayley theorem (the existence of non-trivial functions f annihilating the operator $T: f(T) = \mathbb{O}$) holds, and theorems on unconditionally convergent spectral expansions for operators with a simple point spectrum. The first of these questions is included in a natural way in the framework of the "scalar" theory, because it leads essentially to a study of the maximal functional calculus $f \mapsto f(T)$ for the class of all bounded analytic functions f in the disk. Several Lectures will be devoted to the particulars of this calculus. The general problem of bases of eigenspaces is included for two reasons. It admits a very transparent solution in the language of the characteristic functions and it demonstrates the still unsurmountable difficulties of the theory of operator (matrix) analytic functions with non-commutative values, the principal one being how to adequately translate "additive" problems into the "multiplicative" language of characteristic functions.

The book is divided into 11 Lectures and 5 Appendices. Most of the Lectures are divided vertically: a main canonical part dedicated to the one dimensional model plus a part entitled "Supplements and Bibliographical Notes" developing the first part in various directions. The main part is thereby included in the framework of the Hardy space H^2 and has the character of a special "spectral function theory". The second part usually goes far beyond this framework. The first part is written in great detail and, as far as possible, in an elementary fashion, while the second one is far more condensed and sketchy. Each Lecture concludes with a brief review of the literature and hints to unsolved questions in the portion of the theory in question. What the Appendices 1–5 are about is clear from a glance at the table of contents.

The embryo of this book was the text of lectures I delivered between 1967 and 1972 to an audience of mathematics students at the University of Leningrad. The whole framework has now been modernized, including also the latest achievements in the topics touched upon, some results even being published for the first time. To be able to read the book formally one needs only the standard material in mathematical analysis taught in undergraduate courses (measure theory, Fourier series in L^2, analytic and harmonic functions, the Fourier transform, linear algebra, Banach spaces, the Stone-Weierstrass theorem, the spectral theorem). A number of more special theorems, which are constantly used in the book (and as a rule concern the theory of the Hardy classes H^p), are proved in Appendix 2. The notation employed is usually explained only at the moment of its first appearance[1]; the sign □ stands for the end of a proof or an argument; $\operatorname{clos}_X A$ denotes the closure of the set A in the space X; $\bigvee(\dots)$ the closed linear hull of the set $(\dots)$; Lat T the set of all subspaces invariant with respect to the transformation T.

While working on the book I profited from the support and advice of a great number of persons and to all of them – from the very first listeners to the most severe critics among the specialists – I hereby express my gratitude. With regard to mathematical ideas which have influenced various parts of the book, I owe much to V. I. Vasyunin, S. A. Vinogradov and S. V. Hruščev. I have also received invaluable advice in the final polishing of the text by V. I. Vasyunin, many of whose methodological ideas have been used in the definitive version. Many essential remarks and a lot of friendly assistance came from V. P. Havin, who also read the entire manuscript. I received much further help from V. I. Vasyunin, L. N. Dovbyš, S. V. Kislyakov, V. V. Peller, L. G. Hanin, S. V. Hruščev in composing the necessary technical apparatus connected with the manuscript (bibliography, indices etc.) and likewise I must mention I. V. Bykova, E. B. Eryhova and L. P. Vinogradova for their selfdenying work in preparing the typescript. To all of them I extend my sincere thanks. And finally, but perhaps really in the first place, I am grateful to my family for the encouragement and the enthusiasm with which they endure all the hardships of a mathematical life.

[1] Cf. also the List of Symbols.

Preface to the English Edition

I am very pleased that Springer-Verlag has decided to translate my book thereby making it more readily available to Western readers. I would like to express here my gratitude to the Editors for their goodwill and readiness in taking into account all my requests.

This edition is very significantly enlarged, as compared to the Russian one. Besides smaller local corrections and improvements let me mention here the following additions and changes.

Section 4 of Lecture II has been revised in the spirit of the paper Vasyunin-Nikol'skiĭ (Izv. Akad. Nauk SSSR Ser. Mat. **47**, 942–960 (1983).

Lecture VIII (Section 4) contains now a new approach to the problem of free interpolation in H^∞ (the interpolation operator of Jones-Vinogradov).

There are also four entirely new appendices (Appendix 2–5).

Appendix 2 contains proofs of all assertions about Hardy classes used throughout the book. This makes the book accessible to everyone with just rudimentary background in Analysis.

Appendix 3 contains the modern proof of the Carleson Corona Theorem and its operator theoretic generalization (Tom Wolff's proof as modified by Tolokonnikov et al., with the best estimates for the "solving functions" known at present (1983)).

Appendix 4 gives a rather extensive survey of the theory of Toeplitz and Hankel operators connected with the general orientation of the book – the spectral function theory. I have tried to collect here practically all known results of the theory pertaining to the above orientation. Many of these results are given new proofs and some are published for the first time.

S. V. Hruščev and V. V. Peller have had the kindness to write Appendix 5 specially for this edition. It contains results on singular numbers of Hankel operators and their application to the theory of random processes. It further gives all preliminaries on random processes which are necessary for the proper understanding of the subject.

I would like to thank some of my colleagues for helping me with useful discussions and other support during my work with the appendices, more precisely V. A. Tolokonnikov (Appendix 3), V. I. Vasyunin (Appendix 3 and 4), V. V. Peller, A. L. Volberg, L. N. Dovbyš, S. V. Hruščev (Appendix 4), and finally V. P. Havin who helped me in a variety of ways. L. N. Dovbyš, D. V. Yakubovič, S. V. Kislyakov, N. G. Makarov, V. V. Peller and B. M. Solomyak have assisted me with the proofreading.

Lastly, I am also deeply grateful to the translator Prof. Jaak Peetre for his thorough work.

Contents