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Linear Functional Analysis

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Linear Functional Analysis

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Translated from the Russian
by I. Tweddle

Contents

Preface

Up to a certain time the attention of mathematicians was concentrated on the study of individual objects, for example, specific elementary functions or curves defined by special equations. With the creation of the method of Fourier series, which allowed mathematicians to work with 'arbitrary' functions, the individual approach was replaced by the 'class' approach, in which a particular function is considered only as an element of some 'function space'. More or less simultaneously the development of geometry and algebra led to the general concept of a linear space, while in analysis the basic forms of convergence for series of functions were identified: uniform, mean square, pointwise and so on. It turns out, moreover, that a specific type of convergence is associated with each linear function space, for example, uniform convergence in the case of the space of continuous functions on a closed interval. It was only comparatively recently that in this connection the general idea of a linear topological space (LTS)[1] was formed; here the algebraic structure is compatible with the topological structure in the sense that the basic operations (addition and multiplication by a scalar) are continuous. Included in this scheme are spaces which, historically, had appeared earlier, namely Fréchet spaces (metric with a complete translation invariant metric), Banach spaces (complete normable) and finally the class which is the most special of all but at the same time the most important for applications, Hilbert spaces, whose topology and geometry are defined in a manner which goes back essentially to Euclid – the assignment of a scalar product of vectors.

Using contemporary formal language we can say that LTSs form a category in which continuous homomorphisms, or continuous linear operators (we usually apply the last term to homomorphisms of a space into itself), serve as morphisms. Specific classical examples of linear operators are differentiation and integration or, in a more general form, differential and integral operators. As a rule, integral operators are continuous but this cannot be said of differential operators. Thanks to the construction of a sufficiently general theory of linear operators it became possible to include the latter case.

The story was repeated at the operator level. At first mathematicians studied individual operators but later on it turned out to be useful and necessary to pass to classes. First and foremost in this connection, multiplication of operators (as a rule non-commutative) went out and algebras of operators appeared. Moreover many natural function spaces are also algebras (commutative, of course: typical multiplication is the usual one, i.e. pointwise, or its Fourier equivalent – convolution on an Abelian group). With regard to topology all these situations are covered by the concept of a topological algebra but with a considerable excess of generality. A satisfactory approach is achieved in the narrower setting of

[1] *Translator's note.* I will use LTSs for the plural and write *an* LTS rather than the correct *a* LTS for the indefinite form since the former reads more smoothly. Other abbreviations of similar type will be treated in the same way.

Banach algebras which have proved to be extremely fruitful in harmonic analysis, in representation theory, in approximation theory and so on.

The development of functional analysis ran its course under the powerful influence of theoretical and mathematical physics. Here we may mention, for example, spectral theory, which evolved from wave mechanics, the technique of generalised functions (or distributions), whose construction and widespread introduction was preceded by the systematic practice of using the δ-function in quantum mechanics, ergodic theory, whose fundamental problems were posed by statistical physics, the investigation of operator algebras in connection with applications to quantum field theory and statistical physics and so on. At the present time the ideas, terminology and methods of functional analysis have penetrated deeply not only into natural science but also into such applied disciplines as numerical mathematics and mathematical economics.

In the introductory volume presented below the classical sources of functional analysis are traced, its basic core is described (with a sufficient degree of generality but at the same time with a series of concrete examples and applications) and its principal branches are outlined. An expanded account of a series of specific sections will be given in the subsequent volumes, while certain questions closely connected with linear functional analysis have already been elucidated in previous volumes of the present series. We only touch fragmentarily upon non-linear aspects.

The general plan of the volume was discussed with R.V. Gamkrelidze and N.K. Nikol'skij and individual topics with A.M. Vershik, E.A. Gorin, M.Yu. Lyubich, A.S. Markus, L.A. Pastur and V.A. Tkachenko. Valuable information on certain questions which are elucidated in the volume was kindly provided to the author by V.M. Borok, Yu.A. Brudnyj, V.M. Kadets, M.I. Kadets, V.Eh. Katsnel'son and Yu.I. Lyubarskij. The author offers profound thanks to all the named individuals. He is also most grateful to Dr. Ian Tweddle for his translation of the work into English.