

# Contents

<b>Preface</b>	<b>xiii</b>
<b>Outline</b>	<b>xv</b>
<b>A Theory</b>	<b>1</b>
<b>0 Boolean Algebraic Logic</b>	<b>3</b>
0.0 INTRODUCTION . . . . .	3
0.1 LOGICAL FORMULAE . . . . .	5
0.1.1 The Empty Set . . . . .	5
0.1.2 An Alphabet for Logic . . . . .	6
0.1.3 Well-Formed Formulae . . . . .	7
0.1.4 Exercises . . . . .	9
0.2 LOGICAL TRUTH AND CONNECTIVES . . . . .	14
0.2.1 Logical Truth . . . . .	14
0.2.2 Logical Connectives . . . . .	15
0.2.3 Truth Tables . . . . .	18
0.2.4 Exercises . . . . .	20
0.3 TAUTOLOGIES AND CONTRADICTIONS . . . . .	21
0.3.1 Examples of Tautologies . . . . .	21
0.3.2 Contradictions . . . . .	26
0.3.3 Exercises . . . . .	27
0.4 OTHER METHODS OF PROOF . . . . .	32
0.4.1 Proofs by Tautologies . . . . .	32
0.4.2 Proofs by Contradictions . . . . .	34
0.4.3 Exercises . . . . .	36
0.5 SYNTHESIS OF LOGICAL FORMULAE . . . . .	36
0.5.1 Design of Logical Formulae with Specified Truth Values . . . . .	36
0.5.2 Simplification by Distributivity and Excluded Middle . . . . .	38
0.5.3 Exercises . . . . .	40
0.6 OTHER CONNECTIVES AND APPLICATIONS . . . . .	41
0.6.1 Other Logical Connectives . . . . .	41
0.6.2 Logical Connectives in Binary Arithmetic . . . . .	44
0.6.3 Exercises . . . . .	45
0.7 SYNTHESIS BY KARNAUGH TABLES . . . . .	46
0.7.1 Karnaugh Tables . . . . .	46
0.7.2 Exercises . . . . .	50
0.8 AN APPLICATION TO CIRCUITS . . . . .	52

0.9	PROJECTS . . . . .	53
<b>1</b>	<b>Logic and Deductive Reasoning</b>	<b>55</b>
1.0	INTRODUCTION . . . . .	55
1.1	PROPOSITIONAL CALCULUS . . . . .	56
1.1.1	Formulae, Axioms, Rules, and Proofs . . . . .	56
1.1.2	Examples of Proofs with Axioms P1 and P2 . . . . .	59
1.1.3	Exercises . . . . .	60
1.2	CLASSICAL IMPLICATIONAL CALCULUS . . . . .	62
1.2.1	Derived Rules: Implications Subject to Hypotheses . . . . .	62
1.2.2	Examples of Proofs of Implicational Theorems . . . . .	65
1.2.3	Exercises . . . . .	67
1.3	PROOFS BY CONTRAPOSITION . . . . .	67
1.3.1	Examples of Proofs with Axiom P3 . . . . .	67
1.3.2	Proofs by Reductio ad Absurdum . . . . .	69
1.3.3	Exercises . . . . .	70
1.4	OTHER CONNECTIVES . . . . .	71
1.4.1	Definitions of Other Connectives . . . . .	71
1.4.2	Examples of Proofs of Theorems with Conjunctions . . . . .	72
1.4.3	Examples of Proofs of Theorems with Equivalences . . . . .	74
1.4.4	Examples of Proofs of Theorems with Disjunctions . . . . .	75
1.4.5	Examples of Proofs with Conjunctions and Disjunctions . . . . .	76
1.4.6	Exercises . . . . .	77
1.5	OTHER FORMS OF DEDUCTIVE REASONING . . . . .	78
1.5.1	Conjunctions of Implications . . . . .	78
1.5.2	Proofs by Cases or by Contradiction . . . . .	80
1.5.3	Exercises . . . . .	81
1.6	PREDICATE CALCULUS . . . . .	82
1.6.1	Predicates, Quantifiers, Free or Bound Variables . . . . .	82
1.6.2	Axioms and Rules for the Predicate Calculus . . . . .	85
1.6.3	Examples of Proofs with the Predicate Calculus . . . . .	86
1.6.4	Exercises . . . . .	88
1.7	INFERENCE WITH QUANTIFIERS . . . . .	88
1.7.1	Proofs with Quantifiers . . . . .	88
1.7.2	Proofs With Universal Quantifiers and Other Connectives . . . . .	90
1.7.3	Proofs with Quantifiers and Other Connectives . . . . .	92
1.7.4	Proofs with Restrictions On a Quantified Variable . . . . .	92
1.7.5	Proofs with More than One Quantified Variable . . . . .	94
1.7.6	Exercises . . . . .	95
1.8	FURTHER ISSUES IN LOGIC . . . . .	96
1.9	PROJECTS . . . . .	96
<b>2</b>	<b>Set Theory</b>	<b>97</b>
2.0	INTRODUCTION . . . . .	97
2.1	SETS AND SUBSETS . . . . .	98
2.1.1	Equality and Extensionality . . . . .	98
2.1.2	The Empty Set . . . . .	101
2.1.3	Subsets and Supersets . . . . .	101
2.1.4	Exercises . . . . .	103
2.2	PAIRING, POWER, AND SEPARATION . . . . .	104
2.2.1	Pairing . . . . .	104

2.2.2	Power Sets . . . . .	106
2.2.3	Separation of Sets . . . . .	108
2.2.4	Exercises . . . . .	109
2.3	UNIONS AND INTERSECTIONS OF SETS . . . . .	110
2.3.1	Unions of Sets . . . . .	110
2.3.2	Intersections of Sets . . . . .	114
2.3.3	Unions and Intersections of Sets . . . . .	117
2.3.4	Exercises . . . . .	121
2.4	CARTESIAN PRODUCTS AND RELATIONS . . . . .	124
2.4.1	Cartesian Products of Sets . . . . .	124
2.4.2	Cartesian Products of Unions and Intersections . . . . .	128
2.4.3	Mathematical Relations . . . . .	130
2.4.4	Exercises . . . . .	133
2.5	MATHEMATICAL FUNCTIONS . . . . .	134
2.5.1	Mathematical Functions . . . . .	134
2.5.2	Images and Inverse Images of Sets by Functions . . . . .	137
2.5.3	Exercises . . . . .	140
2.6	COMPOSITE AND INVERSE FUNCTIONS . . . . .	142
2.6.1	Compositions of Functions . . . . .	142
2.6.2	Injective, Surjective, Bijective, and Inverse Functions . . . . .	143
2.6.3	Exercises . . . . .	147
2.7	EQUIVALENCE RELATIONS . . . . .	147
2.7.1	Reflexive, Symmetric, Transitive, or Anti-Symmetric Relations . . . . .	147
2.7.2	Partitions and Equivalence Relations . . . . .	148
2.7.3	Exercises . . . . .	151
2.8	ORDERING RELATIONS . . . . .	152
2.8.1	Preorders and Partial Orders . . . . .	152
2.8.2	Total Orders and Well-Orderings . . . . .	154
2.8.3	Exercises . . . . .	156
2.9	PROJECTS . . . . .	158
<b>3</b>	<b>Induction, Recursion, Arithmetic, Cardinality</b> . . . . .	<b>159</b>
3.0	INTRODUCTION . . . . .	159
3.1	MATHEMATICAL INDUCTION . . . . .	159
3.1.1	The Axiom of Infinity . . . . .	159
3.1.2	The Principle of Mathematical Induction . . . . .	162
3.1.3	Definitions by Mathematical Induction or Recursion . . . . .	163
3.1.4	Exercises . . . . .	165
3.2	ARITHMETIC WITH NATURAL NUMBERS . . . . .	166
3.2.1	Addition with Natural Numbers . . . . .	166
3.2.2	Multiplication with Natural Numbers . . . . .	167
3.2.3	Exercises . . . . .	170
3.3	ORDERS AND CANCELLATIONS . . . . .	171
3.3.1	Orders on the Natural Numbers . . . . .	171
3.3.2	Laws of Arithmetic Cancellations . . . . .	175
3.3.3	Exercises . . . . .	177
3.4	INTEGERS . . . . .	178
3.4.1	Negative Integers . . . . .	178
3.4.2	Arithmetic with Integers . . . . .	181
3.4.3	Order on the Integers . . . . .	183
3.4.4	Non-Negative Integral Powers of Integers . . . . .	187

3.4.5	Exercises . . . . .	188
3.5	RATIONAL NUMBERS . . . . .	189
3.5.1	Definition of Rational Numbers . . . . .	189
3.5.2	Arithmetic with Rational Numbers . . . . .	191
3.5.3	Notation for Sums and Products . . . . .	195
3.5.4	Order on the Rational Numbers . . . . .	199
3.5.5	Exercises . . . . .	201
3.6	FINITE CARDINALITY . . . . .	201
3.6.1	Equal Cardinalities . . . . .	201
3.6.2	Finite Sets . . . . .	204
3.6.3	Exercises . . . . .	208
3.7	INFINITE CARDINALITY . . . . .	208
3.7.1	Infinite Sets . . . . .	208
3.7.2	Denumerable Sets . . . . .	209
3.7.3	The Bernstein–Cantor–Schröder Theorem . . . . .	212
3.7.4	Other Infinite Sets . . . . .	214
3.7.5	Further Issues in Cardinality . . . . .	215
3.7.6	Exercises . . . . .	216
3.8	ARITHMETIC IN FINANCE . . . . .	217
3.8.1	Introduction . . . . .	217
3.8.2	Percentages and Rates . . . . .	217
3.8.3	Sales and Income Taxes . . . . .	217
3.8.4	Compounded Interest . . . . .	218
3.8.5	Loans, Mortgages, and Savings Plans . . . . .	219
3.8.6	Perpetuities . . . . .	220
3.8.7	Exercises . . . . .	220
3.9	PROJECTS . . . . .	221
<b>4</b>	<b>Decidability and Completeness</b> . . . . .	<b>223</b>
4.0	INTRODUCTION . . . . .	223
4.1	LOGICS FOR SCIENTIFIC REASONING . . . . .	224
4.1.1	Scientific Reasoning . . . . .	224
4.1.2	Hypothesis Testing . . . . .	225
4.1.3	Brouwer & Heyting’s Intuitionistic Logic . . . . .	226
4.1.4	Kolmogorov & Johansson’s Minimal Propositional Calculus . . . . .	228
4.1.5	Exercises . . . . .	230
4.2	INCOMPLETENESS . . . . .	231
4.2.1	Tautologies and Theorems . . . . .	231
4.2.2	Incompleteness of the Implicational Calculus . . . . .	232
4.2.3	Exercises . . . . .	234
4.3	LOGICS NOT AMENABLE TO TRUTH TABLES . . . . .	236
4.3.1	Logics with Any Number of Values . . . . .	236
4.3.2	Practical Logics Not Amenable to Truth Tables . . . . .	237
4.3.3	Exercises . . . . .	239
4.4	AUTOMATED THEOREM PROVING . . . . .	240
4.4.1	The Deduction Theorem . . . . .	240
4.4.2	Example: Law of Assertion from the Deduction Theorem . . . . .	242
4.4.3	The Provability Theorem . . . . .	244
4.4.4	The Completeness Theorem . . . . .	246
4.4.5	Example: Peirce’s Law from the Completeness Theorem . . . . .	246
4.4.6	Exercises . . . . .	248

4.5	TRANSFINITE METHODS . . . . .	250
4.5.1	Transfinite Induction . . . . .	250
4.5.2	Transfinite Construction . . . . .	251
4.5.3	Exercises . . . . .	252
4.6	TRANSITIVE SETS AND ORDINALS . . . . .	253
4.6.1	Transitive Sets . . . . .	253
4.6.2	Ordinals . . . . .	254
4.6.3	Well-Ordered Sets of Ordinals . . . . .	255
4.6.4	Unions and Intersections of Sets of Ordinals . . . . .	256
4.6.5	Exercises . . . . .	257
4.7	REGULARITY OF WELL-FORMED SETS . . . . .	258
4.7.1	Well-Formed Sets . . . . .	258
4.7.2	Regularity . . . . .	259
4.7.3	Exercises . . . . .	260
4.8	FURTHER ISSUES IN DECIDABILITY . . . . .	261
4.9	PROJECTS . . . . .	262
<b>B Applications</b>		<b>263</b>
<b>5</b>	<b>Number Theory and Codes</b>	<b>265</b>
5.0	INTRODUCTION . . . . .	265
5.1	THE EUCLIDEAN ALGORITHM . . . . .	265
5.1.1	Division With Integers . . . . .	265
5.1.2	Greatest Common Divisors . . . . .	267
5.1.3	Exercises . . . . .	270
5.2	DIGITAL EXPANSION AND ARITHMETIC . . . . .	271
5.2.1	Expansion of Integers in Powers of an Integral Base . . . . .	271
5.2.2	Digital Integer Arithmetic . . . . .	273
5.2.3	Exercises . . . . .	275
5.3	PRIME NUMBERS . . . . .	276
5.3.1	Prime Numbers . . . . .	276
5.3.2	Prime-Number Factorization . . . . .	277
5.3.3	Primality Testing . . . . .	279
5.3.4	Exercises . . . . .	279
5.4	MODULAR ARITHMETIC . . . . .	280
5.4.1	Modular Integers . . . . .	280
5.4.2	Modular Addition . . . . .	282
5.4.3	Modular Multiplication . . . . .	284
5.4.4	Modular Division . . . . .	285
5.4.5	Divisibility Tests . . . . .	287
5.4.6	Exercises . . . . .	289
5.5	MODULAR CODES . . . . .	290
5.5.1	The International Standard Book Number (ISBN) Code . . . . .	290
5.5.2	The Universal Product Code (UPC) . . . . .	291
5.5.3	The Bank Identification Code . . . . .	291
5.5.4	The U.S. postal nine-digit ZIP Code . . . . .	292
5.5.5	Exercises . . . . .	292
5.6	EUCLID, EULER, & FERMAT'S THEOREMS . . . . .	294
5.6.1	Fermat's Little Theorem . . . . .	294
5.6.2	The Euler–Fermat Theorem . . . . .	295

5.6.3	Exercises . . . . .	296
5.7	RIVEST–SHAMIR–ADLEMAN (RSA) CODES . . . . .	297
5.7.1	The Rivest–Shamir–Adleman (RSA) Public-Key . . . . .	297
5.7.2	Proof of the Correctness of the RSA Algorithm . . . . .	298
5.7.3	Exercises . . . . .	300
5.8	FURTHER ISSUES IN NUMBER THEORY . . . . .	301
5.9	PROJECTS . . . . .	302
<b>6</b>	<b>Ciphers, Combinatorics, and Probabilities</b>	<b>303</b>
6.0	INTRODUCTION . . . . .	303
6.1	ALGEBRA . . . . .	304
6.1.1	Definitions and Examples of Mathematical Groups . . . . .	304
6.1.2	Existence and Uniqueness of Identities and Inverses . . . . .	305
6.1.3	Bijections of Groups . . . . .	308
6.1.4	Exercises . . . . .	311
6.2	PERMUTATIONS . . . . .	313
6.2.1	Permutations and Orbits . . . . .	313
6.2.2	Transpositions . . . . .	316
6.2.3	Exercises . . . . .	319
6.3	CYCLIC PERMUTATIONS . . . . .	320
6.3.1	Cycles . . . . .	320
6.3.2	Factorization of Permutations by Cycles . . . . .	322
6.3.3	Exercises . . . . .	323
6.4	SIGNATURES OF PERMUTATIONS . . . . .	324
6.4.1	Even and Odd Permutations . . . . .	324
6.4.2	Signatures of Composite Permutations . . . . .	326
6.4.3	Exercises . . . . .	328
6.5	ARRANGEMENTS . . . . .	328
6.5.1	Arrangements Without Repetition . . . . .	328
6.5.2	Arrangements With Repetitions . . . . .	329
6.5.3	Exercises . . . . .	330
6.6	COMBINATIONS . . . . .	331
6.6.1	Combinations Without Repetition . . . . .	331
6.6.2	Combinations With Repetition . . . . .	335
6.6.3	Permutations With Repetition . . . . .	336
6.6.4	Exercises . . . . .	339
6.7	PROBABILITY . . . . .	339
6.7.1	Probability Spaces . . . . .	339
6.7.2	The Exclusion-Inclusion Principle . . . . .	343
6.7.3	Conditional Probabilities . . . . .	344
6.7.4	Applications of Binomial Probabilities . . . . .	347
6.7.5	Exercises . . . . .	350
6.8	THE ENIGMA MACHINES . . . . .	351
6.8.1	Design and Operation of the ENIGMA Machines . . . . .	351
6.8.2	Breaking the ENIGMA Ciphers . . . . .	354
6.8.3	Reconstructing the ENIGMA Wiring . . . . .	355
6.8.4	Exercises . . . . .	357
6.9	PROJECTS . . . . .	359

<b>7</b>	<b>Graph Theory</b>	<b>361</b>
7.0	INTRODUCTION	361
7.1	MATHEMATICAL GRAPHS	362
7.1.1	Directed Graphs	362
7.1.2	Undirected Graphs	363
7.1.3	Degrees of Vertices	366
7.1.4	Exercises	368
7.2	PATH-CONNECTED GRAPHS	369
7.2.1	Walks in Graphs	369
7.2.2	Path-Connected Components of Graphs	371
7.2.3	Longer Walks	373
7.2.4	Longest Paths	374
7.2.5	Exercises	375
7.3	EULER AND HAMILTONIAN CIRCUITS	375
7.3.1	Circuits	375
7.3.2	Hamiltonian Paths and Circuits	376
7.3.3	Eulerian Paths and Circuits	378
7.3.4	Exercises	379
7.4	MATHEMATICAL TREES	380
7.4.1	Trees	380
7.4.2	Spanning Trees	380
7.4.3	Directed Trees and Star-Shaped Graphs	381
7.4.4	Exercises	382
7.5	WEIGHTED GRAPHS AND MULTIGRAPHS	382
7.5.1	Weighted Graphs	382
7.5.2	Shortest Path and Minimum-Weight Paths or Trees	383
7.5.3	Maximal Flows in Transportation Graphs	385
7.5.4	Exercises	387
7.6	BIPARTITE GRAPHS AND MATCHINGS	388
7.6.1	Bipartite Graphs	388
7.6.2	Maximum Matching	390
7.6.3	Exercises	392
7.7	GRAPH THEORY IN SCIENCE	392
7.7.1	The Shape of Molecules	392
7.7.2	The Shape of Hydrocarbons	393
7.7.3	Sequences of Radioactive Decays	395
7.7.4	Exercises	396
7.8	FURTHER ISSUES: PLANAR GRAPHS	397
7.9	PROJECTS	398
	<b>Bibliography</b>	<b>399</b>
	<b>Index</b>	<b>405</b>