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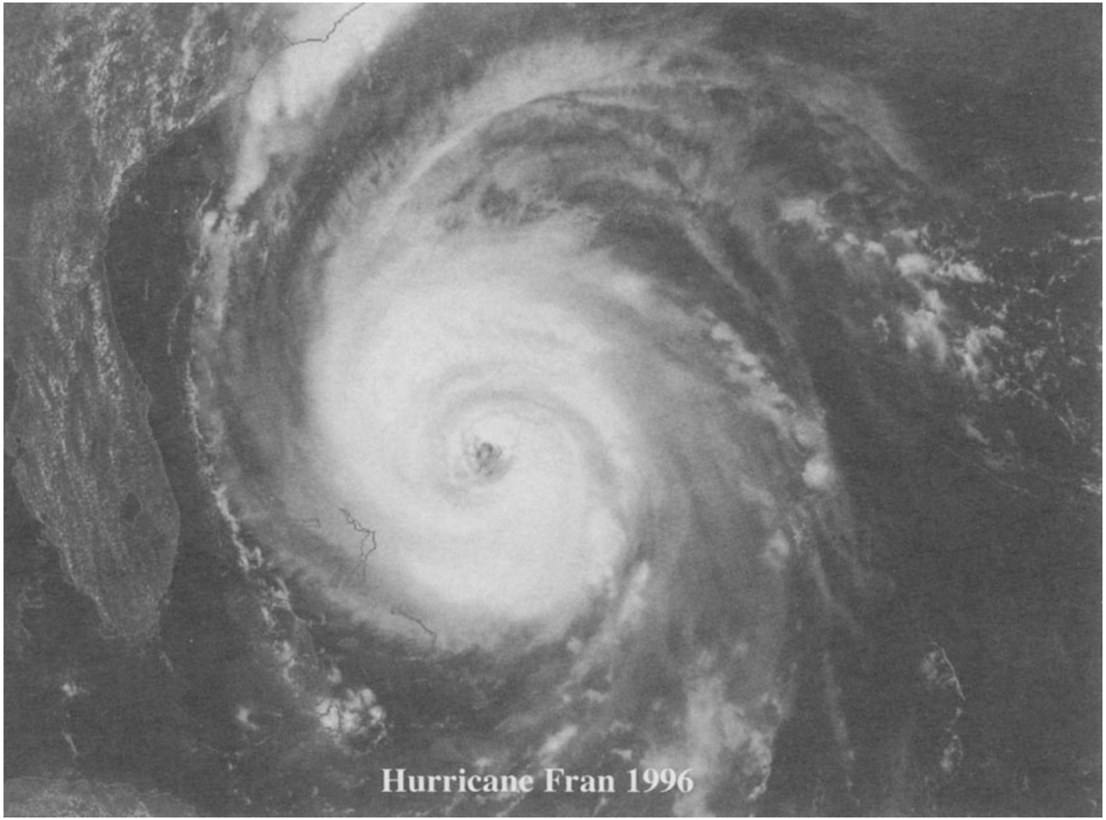
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Paul K. Newton

The N-Vortex Problem

Analytical Techniques

With 79 Figures

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This book is dedicated
with all my love
to
Lynn and Daniel.

Preface

This book is an introduction to current research on the N-vortex problem of fluid mechanics in the spirit of several works on N-body problems from celestial mechanics, as for example Pollard (1966), Szebehely (1967), or Meyer and Hall (1992). Despite the fact that the field has progressed rapidly in the last 20 years, no book covers this topic, particularly its more recent developments, in a thorough way. While Saffman's *Vortex Dynamics* (1992) covers the general theory from a classical point of view, and Marchioro and Pulvirenti's *Mathematical Theory of Incompressible Nonviscous Fluids* (1994), Doering and Gibbon's *Applied Analysis of the Navier–Stokes Equations* (1995), and Majda and Bertozzi's *Vorticity and Incompressible Fluid Flow* (2001) cover much of the relevant mathematical background, none of these discusses the more recent literature on integrable and nonintegrable point vortex motion in any depth. Chorin's *Vorticity and Turbulence* (1996) focuses on aspects of vorticity dynamics that are most relevant toward an understanding of turbulence, while Arnold and Khesin's *Topological Methods in Hydrodynamics* (1998) lays the groundwork for a geometrical and topological study of the Euler equations. Ottino's *The Kinematics of Mixing: Stretching, Chaos, and Transport* (1989) is an introductory textbook on the use of dynamical systems techniques in the study of fluid mixing, while Wiggins' *Chaotic Transport in Dynamical Systems* (1992) describes techniques that are of general use, without focusing specifically on vortex motion. All of these books cover aspects of the topics I treat, but none focuses on exactly the same issues.

My goal is to describe the Hamiltonian aspects of vortex dynamics so that graduate students and researchers can use the book as an entry point into the rather large literature on integrable and nonintegrable vortex problems. By describing the N-vortex problem as a branch of dynamical systems theory in the way the N-body problem of celestial mechanics is often presented, I have tried to keep my focus fairly narrow, but deeper than a broader literature survey would be. Despite the fact that problems in celestial mechanics have historically been among the key driving forces behind progress in dynamics, [1] research in this area has fallen out of favor and celestial mechanics is seldom taught in most physics, engineering, and mathematics departments these days. By contrast, the field of vortex dynamics is lively and active, using techniques that have widespread applicability to many general problems in dynamics and modern applied mathematics. These include integrable and nonintegrable Hamiltonian methods, nonlinear stability techniques for fixed and relative equilibria, long-time existence theory, finite-time blow-up (collisions), geometric phase space methods including KAM and singular perturbation theory, geometric phases, transport and mixing, numerical methods and convergence theories relating discrete approximations to their continuum limits, topology and the theory of knots, geometry of curves and surfaces, Lie groups and Lie algebras, statistical mechanics, ergodic theory, and probabilistic methods, to name just a few. While I have not covered all of those topics, my belief is that an extensive study of the N-vortex problem provides an ideal entry into the field of nonlinear dynamics that is physically relevant and mathematically rich.

In the first chapter I present an overview of the two main themes of the book, vorticity dynamics and Hamiltonian systems. The chapter is written more in a review style than are the subsequent ones and, hence, serves the main purpose of introducing many of the background topics necessary for understanding topics covered more thoroughly later in the book. At the end of the chapter, I formulate some of the key questions that will be covered in the chapters that follow. Chapter 2 covers much of what is known concerning the N-vortex problem in the plane with no boundaries. I describe in detail the integrable 3-vortex problem following the classical works of Kirchhoff (1876), Gröbli (1877), Synge (1949), Novikov (1978), and Aref (1979). The 3-vortex problem is fundamental to much of the development of the subject. There are many reasons for this, not the least of which is the fact that the interactions of the general N-vortex problem can

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[1]The evidence of this can, of course, be found starting with Newton's *Philosophiae Naturalis Principia Mathematica* (1687), Lagrange's *Mécanique Analytique* (1788), Laplace's *Traité de Mécanique Céleste* (1799), Tisserand's *Traité de Mécanique Céleste* (1889), Poincaré's *Les Méthodes Nouvelles de la Mécanique Céleste* (1892), Whittaker's *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies* (1904), Moulton's *An Introduction to Celestial Mechanics* (1902), Birkhoff's *Dynamical Systems* (1927), Wintner's *Analytical Foundations of Celestial Mechanics* (1941), Brouwer and Clemence's *Methods of Celestial Mechanics* (1961), and Moser and Siegel's *Lectures on Celestial Mechanics* (1971). A beautiful, recent book describing part of this history is Barrow-Green (1997).

be written as interacting triads; there are analogues of this decomposition for general nonlinear oscillators and for turbulent wave interactions, where triad dynamics plays a central role. I mention also the fact that the 3-vortex problem is the simplest problem (in terms of smallest N) capable of generating new length and time scales through its dynamics. For the 4-vortex problem I describe the coordinates introduced in Khanin (1982) and Lim (1991a), which conveniently reduce the system from four to two degrees of freedom. I then give a proof of nonintegrability of the restricted 4-vortex problem using Melnikov theory. The original proof is found in Ziglin (1980), followed by Koiller and Carvalho (1985, 1989). However, I follow in more detail the recent proof given in Castilla, Moauro, Negrini, and Oliva (1993).

Chapter 3 contains a discussion of vortex problems in the plane with boundaries. I begin by recalling some classical results from potential theory and the construction of Green's functions for closed domains. Methods for explicitly constructing the relevant Hamiltonian in a closed domain are presented, such as the method of images and conformal mapping techniques. I then explain how integrability is broken in a closed domain without symmetries, following the work of Zannetti and Franchessi (1995). In Chapter 4, I introduce the N-vortex problem on a spherical shell (the one-layer model). The 3-vortex problem is described in detail, following Kidambi and Newton (1998, 1999, 2000a, 2000b). I describe the integrability of this problem, classification of equilibria, as well as the nonequilibrium process of 3-vortex spherical collapse, and contrast the differences between spherical collapse and planar collapse. I discuss what is known about the possible instantaneous streamline topologies that are allowable on the sphere, along with a general classification of topologies for the 3-vortex problem and how this might be relevant in the wider context of atmospheric flows and the "nonlinear decomposition" of weather patterns. I also introduce techniques to treat the case with solid boundaries on a spherical surface, following the recent work of Kidambi and Newton (2000b).

Chapter 5 contains a discussion of geometric phases for vortex problems in the plane, following Newton (1994), and Shashikanth and Newton (1998, 1999). Their role in determining the growth rate of spiral interfaces in vortex-dominated flows is described in the context of several prototypical configurations. In Chapter 6 I present an overview of the statistical mechanics treatment of point vortex motion, which was initiated in a famous paper of Onsager (1949). More recent treatments are described, including those of Pointin and Lundgren (1976), Lundgren and Pointin (1977a,b), Miller and Robert (1991), Miller (1990), Montgomery and Joyce (1974), Robert (1991), Robert and Sommeria (1991), and Lim (1998a, 1999).

Chapters 7 and 8 deal with extensions to the basic theory. In Chapter 7 I relax the assumption that the vorticity is concentrated at singular points and hence describe the dynamics associated with vortex patches in the plane. I explain the work of Kida (1981), in which an exact elliptical patch is derived in the presence of time-independent background strain and vorticity. This work is generalized in the paper of Neu (1984), which is also described. I then present the so-called *moment model* of Melander, Zabusky, and Styczek (1986) in which a self-consistent Hamiltonian system is derived for the interaction of vortex patches under the assumption that the patches are nearly circular and well separated. I also describe a shear layer model introduced and analyzed in Meiburg and Newton (1991) and Newton and Meiburg (1991) based on a viscously decaying, spatially periodic row of vortices. Chapter 8 treats vortex filament dynamics in three dimensions. First, I describe the localized induction equation for the evolution of a thin isolated filament. The DaRios–Betchov equations governing the evolution of the curvature and torsion of the filament are derived, and special solutions are discussed, such as circular rings, helices, torus knots, and solitary wave loops traveling on a filament. Hasimoto's transformation of the localized induction equation to the integrable nonlinear Schrödinger equation is described in some detail, along with discussion of some of the known invariants based on this approach. Higher order theories that include vortex core structure and self-stretching mechanisms are described and a simple model of interacting nearly parallel filaments is presented, following the work of Klein, Majda, and Damodaran (1995). I end with a brief description of the so-called "vorton model," which has been used recently, mostly for numerical calculations. Hence, the book naturally divides into two parts, the first being Chapters 1 through 3, which present much of the basic two-dimensional point vortex theory that is now considered classical. More recent applications and extensions of the basic theory are covered in Chapters 4 through 8, which form the second part of the book.

Despite my best efforts at presenting a comprehensive treatment, there are still many topics I have been forced to ignore, most notably the rather large literature on all numerical aspects associated with vortex dynamics. This includes all the work involving convergence of point vortex decompositions to their continuum Euler limit, which I only briefly summarize. Readers interested in these topics can read the recent book of Cottet and Koumoutsakos (2000). In addition, there are many influential papers on the subject that are primarily computational, which I have not emphasized. In the end I have chosen the topics that I hope will be most useful to the nonlinear dynamicist interested in learning analytical techniques. I have tried to keep the book as self-contained as possible, so that the only training required of the reader is a good background in advanced calculus and ordinary and partial differential equations at the level of a typical undergraduate engineering, physics, or applied mathematics major. An introductory graduate course in fluid mechanics and differential equations would be quite useful, but could be picked up as needed by the diligent reader. Since my experience has been that students come to this subject with many different backgrounds, ultimately, the most important prerequisite is a desire to learn the topics described. At the end of each chapter there are exercises of varying difficulty which in many cases require the reader to fill in details of proofs or complete examples whose inclusion would otherwise make the book too long and tedious. Completing these exercises should help the reader compensate for his or her incomplete background.

By and large, I have avoided extensive discussions of the classical literature of the type found in other books such as Lamb (1932), Batchelor (1967), or Saffman (1992). References are made to the relevant papers, but it is assumed that the reader will go to the original sources as needed for detailed derivations of classical theorems and special solutions. In addition, I have tried to focus on models that strike a balance between simplicity and faithfulness to physical reality, describing results that are explicit, sometimes at the expense of generality. My goal is to present tractable models where one can learn physics along with techniques of nonlinear dynamics that can be used in other contexts. In all cases I have tried to clearly explain the approximations being used and the subsequent shortcomings of the model being described.

There are many open problems associated with N-vortex motion; I mention some that are interesting:

- It would be nice to obtain rigorous estimates for the diffusion rate of particle motion in flows populated by vortices, particularly in closed domains or on compact surfaces, such as spheres. Results of this type would be relevant to fluid mixing and transport in turbulent flows. It would be particularly interesting to relate the global geometric properties of the instantaneous streamline patterns associated with the velocity field to the statistical properties of particle transport, the key being an analysis of the particle distribution function with a velocity field generated by point vortices. Arnold diffusion presumably also plays a role here, but there are few analytical results in this direction.

- Related to the previous problem, rigorous results on the mixing properties of vortex motion in compact regions and on surfaces of general curvature would be important; for example, under what conditions the motion is ergodic, mixing, Bernouilli ... Techniques borrowed from billiard dynamics where similar issues arise might well be useful.

- The general formulation of non-adiabatic geometric phases for vortex motion is interesting and more general than in the adiabatic context. Results along these lines could well be important for developing a control theory based on vortex motion as the importance of geometric phases in the control of robotic systems and in other related contexts is currently being fleshed out.

- Understanding the dynamics of point vortices on a sphere with additional effects, such as rotation and vertical density stratification, is crucial for geophysical applications where one is interested in large-scale atmospheric and oceanographic mixing. Vertical density stratification is sometimes treated with N-layer models where vortices interact within each layer, as well as across layers, but not much has been done analytically with these models as compared to what has been done in one layer.

- A general study of the N-vortex collision problem has not yet been carried out. For instance, we do not know if there are any non self-similar collisions of vortices in the plane or on the sphere, and vortex collisions in domains with boundaries are only beginning to be investigated. In general, much more is known about N-body collisions in the context of celestial mechanics, and some of these techniques might prove useful.

- A classification and understanding of all possible instantaneous streamline patterns that are allowable in the plane and on the sphere for an N-vortex problem would be relevant to an understanding of mixing processes. Understanding the dynamical transition from one pattern to another could prove important for these purposes, and further results are needed on the role that these finite-time structures play in the mixing process. A general program for topologically classifying all integrable Hamiltonian systems is at the present time being developed by several groups. The manner in which the N-vortex problem fits within these classifications is currently unclear.

- A general control theory based on knowledge of the vortex motion would be desirable. Controlling the vortex motion by passive or active means near boundaries could have important applications in many technological and environmental flows (for issues like noise supression), and the further development of a Lagrangian theory might prove advantageous over a Hamiltonian one.

- Developing techniques for understanding vortex motion in the far from integrable regime, where the number of vortices is large, but not large enough to warrant the use of statistical mechanics, would be worthwhile. This intermediate regime remains largely unexplored.

- The equations for the interaction of three-dimensional vortex filaments that are nearly parallel have recently been derived as a coupled Schrödinger system (see Chapter 8) and have yet to be thoroughly explored. It would be particularly interesting to analyze the case of 3-vortex filament collapse, using what is known about the collapsing system when the filaments remain perfectly parallel. In addition, using the coupled system as a starting point, one could study the effect of the third dimension on the mixing and transport of particles as they migrate out of plane.

I would like to thank many people who have helped me in learning and presenting the material in this book. Larry Sirovich and Joe Keller were my first teachers of fluid dynamics. They showed me the beauty of the subject and introduced me to some of its challenges. I have learned many things about the Hamiltonian aspects of the field from Jerry Marsden and I thank him for his encouragement and steady advice on all aspects of this project.

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[2] Downloadable lectures are available at http://online.itp.ucsb.edu/online/hydrot00/

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