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Invariant Theory of Finite Groups

**Mara D. Neusel
Larry Smith**



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ABSTRACT. This book gives a comprehensive overview of the invariant theory of finite groups acting linearly on polynomial algebras. It spans the gamut from the classical methods and results of Emmy Noether, T. Molien, D. Hilbert and L. E. Dickson to the modern methods and insights obtained from using Steenrod algebra technology, as in the proof of the Landweber-Stong conjecture. Numerous examples illustrate the theory and techniques introduced.

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