

---

---

# Mathematical Methods in Image Reconstruction

*Frank Natterer*

*Frank Wübbeling*

Universität Münster  
Münster, Germany

**siam**

Society for Industrial and Applied Mathematics  
Philadelphia

Copyright © 2001 by the Society for Industrial and Applied Mathematics.

10 9 8 7 6 5 4 3 2

All rights reserved. Printed in the United States of America. No part of this book may be reproduced, stored, or transmitted in any manner without the written permission of the publisher. For information, write to the Society for Industrial and Applied Mathematics, 3600 University City Science Center, Philadelphia, PA 19104-2688.

### **Library of Congress Cataloging-in-Publication Data**

Mathematical methods in image reconstruction / Frank Natterer...[et al.]

p. cm. - (SIAM monographs on mathematical modeling and computation)

Includes bibliographical references and index.

ISBN-10: 0-89871-622-5 (pbk.)

ISBN-13: 978-0-898716-22-1 (pbk.)

1. Image processing -Congresses. I. Natterer, F. (Frank), 1941- II. Series.

TA1637 .M356 2001

621.36'7--dc21

00-053804

# Contents

<b>Preface</b>	<b>ix</b>
<b>List of Symbols</b>	<b>xi</b>
<b>1 Introduction</b>	<b>1</b>
1.1 The Basic Example . . . . .	1
1.2 Overview . . . . .	2
1.3 Mathematical Preliminaries . . . . .	3
1.3.1 Fourier analysis . . . . .	3
1.3.2 Some integral operators . . . . .	5
1.3.3 The Moore–Penrose generalized inverse . . . . .	5
1.3.4 The singular value decomposition . . . . .	5
1.3.5 Special functions . . . . .	6
1.3.6 The fast Fourier transform . . . . .	8
<b>2 Integral Geometry</b>	<b>9</b>
2.1 The Radon Transform . . . . .	9
2.2 The Ray Transform . . . . .	17
2.3 The Cone Beam Transform . . . . .	23
2.4 Weighted Transforms . . . . .	27
2.4.1 The attenuated ray transform . . . . .	27
2.4.2 The Feig–Greenleaf transform . . . . .	30
2.4.3 The windowed ray transform . . . . .	31
2.5 Integration over Curved Manifolds . . . . .	31
2.5.1 Computing an even function on $S^2$ from its integrals over equatorial circles . . . . .	32
2.5.2 Reduction of problems on the sphere to the Radon transform . . . . .	33
2.5.3 Reconstruction from spherical averages . . . . .	34
2.5.4 More general manifolds . . . . .	36
2.6 Vector Fields . . . . .	36
<b>3 Tomography</b>	<b>41</b>
3.1 Transmission Tomography . . . . .	41
3.1.1 Parallel scanning geometry . . . . .	41

3.1.2	Fan beam scanning geometry . . . . .	42
3.1.3	3D helical scanning . . . . .	43
3.1.4	3D cone beam scanning . . . . .	43
3.2	Emission Tomography . . . . .	44
3.3	Diffraction Tomography . . . . .	46
3.4	Magnetic Resonance Imaging . . . . .	50
3.5	Electron Tomography . . . . .	54
3.6	Radar . . . . .	55
3.6.1	Synthetic aperture radar . . . . .	55
3.6.2	Range–Doppler radar . . . . .	56
3.7	Vector Tomography . . . . .	57
3.7.1	Doppler tomography . . . . .	57
3.7.2	Schlieren tomography . . . . .	58
3.7.3	Photoelastic tomography . . . . .	58
3.8	Seismic Tomography . . . . .	59
3.8.1	Travel time tomography . . . . .	59
3.8.2	Reflection tomography . . . . .	59
3.8.3	Waveform tomography . . . . .	60
3.9	Historical Remarks . . . . .	61
<b>4</b>	<b>Stability and Resolution</b>	<b>63</b>
4.1	Stability . . . . .	63
4.2	Sampling . . . . .	65
4.3	Resolution . . . . .	71
4.4	The FFT on Nonequispaced Grids . . . . .	78
<b>5</b>	<b>Reconstruction Algorithms</b>	<b>81</b>
5.1	The Filtered Backprojection Algorithm . . . . .	81
5.1.1	Standard parallel scanning . . . . .	83
5.1.2	Parallel interlaced scanning . . . . .	87
5.1.3	Standard fan beam scanning . . . . .	90
5.1.4	Linear fan beam scanning . . . . .	93
5.1.5	Fast backprojection . . . . .	95
5.1.6	The point spread function . . . . .	96
5.1.7	Noise in the filtered backprojection algorithm . . . . .	97
5.1.8	Filtered backprojection for the exponential Radon transform . . . . .	99
5.1.9	Filtered backprojection for the attenuated Radon transform . . . . .	99
5.2	Fourier Reconstruction . . . . .	100
5.2.1	Standard Fourier reconstruction . . . . .	100
5.2.2	The gridding method . . . . .	102
5.2.3	The linogram algorithm . . . . .	106
5.2.4	Fourier reconstruction in diffraction tomography and MRI . . . . .	108
5.3	Iterative Methods . . . . .	110
5.3.1	ART . . . . .	110
5.3.2	The EM algorithm . . . . .	118
5.3.3	Other iterative algorithms . . . . .	124

---

5.4	Direct Algebraic Algorithms . . . . .	125
5.5	3D Algorithms . . . . .	127
5.5.1	The FDK approximate formula . . . . .	128
5.5.2	Grangeat's method . . . . .	129
5.5.3	Filtered backprojection for the cone beam transform . . . . .	131
5.5.4	Filtered backprojection for the ray transform . . . . .	132
5.5.5	The Radon transform in 3D . . . . .	133
5.6	Circular Harmonic Algorithms . . . . .	134
5.6.1	Standard parallel scanning . . . . .	134
5.6.2	Standard fan beam scanning . . . . .	136
5.7	ART for Nonlinear Problems . . . . .	137
<b>6</b>	<b>Problems That Have Peculiarities</b>	<b>139</b>
6.1	Unknown Orientations . . . . .	139
6.1.1	The geometric method . . . . .	139
6.1.2	The moment method . . . . .	141
6.1.3	The method of Provencher and Vogel . . . . .	142
6.1.4	The 2D case . . . . .	143
6.2	Incomplete Data . . . . .	144
6.2.1	Uniqueness and stability . . . . .	144
6.2.2	Reconstruction methods . . . . .	147
6.2.3	Truncated projections in PET . . . . .	148
6.2.4	Conical tilt problem in electron tomography . . . . .	150
6.3	Discrete Tomography . . . . .	151
6.4	Simultaneous Reconstruction of Attenuation and Activity . . . . .	152
6.5	Local Tomography . . . . .	155
6.6	Few Data . . . . .	159
<b>7</b>	<b>Nonlinear Tomography</b>	<b>161</b>
7.1	Tomography with Scatter . . . . .	161
7.2	Optical Tomography . . . . .	163
7.2.1	The transport model . . . . .	163
7.2.2	The diffusion model . . . . .	165
7.2.3	The linearized problem . . . . .	168
7.2.4	Calderon's method . . . . .	170
7.2.5	The transport-backtransport algorithm . . . . .	172
7.2.6	The diffusion-backdiffusion algorithm . . . . .	174
7.3	Impedance Tomography . . . . .	176
7.4	Ultrasound Tomography . . . . .	178
7.4.1	Frequency domain ultrasound tomography . . . . .	179
7.4.2	Time domain ultrasound tomography . . . . .	185
	<b>Bibliography</b>	<b>189</b>
	<b>Index</b>	<b>209</b>