

# **Complex Analysis in One Variable**

## **Second Edition**

Raghavan Narasimhan  
Yves Nievergelt

Raghavan Narasimhan  
Department of Mathematics  
University of Chicago  
Chicago, IL 60637  
U.S.A.

Yves Nievergelt  
Department of Mathematics  
Eastern Washington University  
Cheney, WA 99004  
U.S.A.

### Library of Congress Cataloging-in-Publication Data

Narasimhan, Raghavan.

Complex analysis in one variable.—2nd ed. / Raghavan Narasimhan and Yves Nievergelt.  
p. cm.

Includes bibliographical references and index.

ISBN 978-1-4612-6647-1 ISBN 978-1-4612-0175-5 (eBook)

DOI 10.1007/978-1-4612-0175-5

1. Functions of complex variables. 2. Mathematical analysis. I. Nievergelt, Yves. II. Title.

QA331.N27 2000  
515'.9—dc21

00-051906  
CIP

---

AMS Subject Classifications: 30-01, 30A05, 30A99, 30D30, 31A05, 32A10

---

Printed on acid-free paper.



© 2001 Springer Science+Business Media New York

Originally published by Birkhäuser Boston in 2001

Softcover reprint of the hardcover 2nd edition 2001

All rights reserved. This work may not be translated or copied in whole or in part without the written permission of the publisher Springer Science+Business Media, LLC, except for brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed is forbidden.

The use of general descriptive names, trade names, trademarks, etc., in this publication, even if the former are not especially identified, is not to be taken as a sign that such names, as understood by the Trade Marks and Merchandise Marks Act, may accordingly be used freely by anyone.

ISBN 978-1-4612-6647-1 SPIN 10749282

Typeset in L<sup>A</sup>T<sub>E</sub>X2<sub>ε</sub> by T<sub>E</sub>Xniques, Inc., Cambridge, MA.

9 8 7 6 5 4 3 2 1

# Contents

<b>Preface to the Second Edition</b>	<b>ix</b>
<b>Preface to the First Edition</b>	<b>xi</b>
<b>Notation and Terminology</b>	<b>xiii</b>
<b>I Complex Analysis in One Variable</b>	
<i>Raghavan Narasimhan</i>	<b>1</b>
<b>1 Elementary Theory of Holomorphic Functions</b>	<b>3</b>
1 Some basic properties of $\mathbb{C}$ -differentiable and holomorphic functions . . . . .	4
2 Integration along curves . . . . .	10
3 Fundamental properties of holomorphic functions . . . . .	22
4 The theorems of Weierstrass and Montel . . . . .	32
5 Meromorphic functions . . . . .	36
6 The Looman–Menchoff theorem . . . . .	43
<b>2 Covering Spaces and the Monodromy Theorem</b>	<b>53</b>
1 Covering spaces and the lifting of curves . . . . .	53
2 The sheaf of germs of holomorphic functions . . . . .	55
3 Covering spaces and integration along curves . . . . .	57
4 The monodromy theorem and the homotopy form of Cauchy’s theorem . . . . .	60
5 Applications of the monodromy theorem . . . . .	63
<b>3 The Winding Number and the Residue Theorem</b>	<b>69</b>
1 The winding number . . . . .	69
2 The residue theorem . . . . .	73
3 Applications of the residue theorem . . . . .	79
<b>4 Picard’s Theorem</b>	<b>87</b>

<b>5</b>	<b>Inhomogeneous Cauchy–Riemann Equation and Runge’s Theorem</b>	<b>97</b>
1	Partitions of unity . . . . .	97
2	The equation $\frac{\partial u}{\partial \bar{z}} = \phi$ . . . . .	99
3	Runge’s theorem . . . . .	103
4	The homology form of Cauchy’s theorem . . . . .	111
<b>6</b>	<b>Applications of Runge’s Theorem</b>	<b>115</b>
1	The Mittag-Leffler theorem . . . . .	115
2	The cohomology form of Cauchy’s theorem . . . . .	119
3	The theorem of Weierstrass . . . . .	121
4	Ideals in $\mathcal{H}(\Omega)$ . . . . .	127
<b>7</b>	<b>Riemann Mapping Theorem and Simple Connectedness in the Plane</b>	<b>139</b>
1	Analytic automorphisms of the disc and of the annulus . . . . .	139
2	The Riemann mapping theorem . . . . .	143
3	Simply connected plane domains . . . . .	145
<b>8</b>	<b>Functions of Several Complex Variables</b>	<b>151</b>
<b>9</b>	<b>Compact Riemann Surfaces</b>	<b>161</b>
1	Definitions and basic theorems . . . . .	161
2	Meromorphic functions . . . . .	166
3	The cohomology group $H^1(\mathcal{U}, \mathcal{O})$ . . . . .	167
4	A theorem from functional analysis . . . . .	171
5	The finiteness theorem . . . . .	176
6	Meromorphic functions on a compact Riemann surface . . . . .	179
<b>10</b>	<b>The Corona Theorem</b>	<b>187</b>
1	The Poisson integral and the theorem of F. and M. Riesz . . . . .	188
2	The corona theorem . . . . .	197
<b>11</b>	<b>Subharmonic Functions and the Dirichlet Problem</b>	<b>209</b>
1	Semi-continuous functions . . . . .	209
2	Harmonic functions and Harnack’s principle . . . . .	212
3	Convex functions . . . . .	215
4	Subharmonic functions: Definition and basic properties . . . . .	219
5	Subharmonic functions: Further properties and application to convexity theorems . . . . .	227
6	Harmonic and subharmonic functions on Riemann surfaces . . . . .	237
7	The Dirichlet problem . . . . .	237
8	The Radó–Cartan theorem . . . . .	244
	<b>Appendix: Baire’s Theorem</b>	<b>253</b>

**II Exercises**

*Yves Nievergelt*

**255**

**Introduction**

**257**

**0 Review of Complex Numbers**

**259**

- 1 Algebraic properties of the complex numbers . . . . . 259
- 2 Complex equations of generalized circles . . . . . 261
- 3 Complex fractional linear transformations . . . . . 262
- 4 Topological concepts . . . . . 265

**1 Elementary Theory of Holomorphic Functions**

**267**

- 1 Some basic properties of  $\mathbb{C}$ -differentiable and holomorphic functions . . . . . 267
  - 1.1 Complex derivatives and Cauchy–Riemann equations . . . . . 267
  - 1.2 Differentials and conformal maps . . . . . 269
  - 1.3 Conformal maps . . . . . 270
  - 1.4 Radius of convergence of power series . . . . . 275
  - 1.5 Exponential, trigonometric, and dilogarithm functions . . . . . 277
- 2 Integration along curves . . . . . 278
  - 2.1 Complex line integrals . . . . . 278
  - 2.2 Complex derivatives of line integrals . . . . . 279
  - 2.3 Remainder of complex Taylor polynomials . . . . . 281
  - 2.4 H. A. Schwarz’s reflection principle . . . . . 281
- 3 Fundamental properties of holomorphic functions . . . . . 282
  - 3.1 The complex exponential function . . . . . 282
  - 3.2 Holomorphic functions . . . . . 284
  - 3.3 Bounds on the size of roots of polynomials . . . . . 285
  - 3.4 Principal branch of the complex square root . . . . . 287
  - 3.5 Complex square roots in celestial mechanics . . . . . 288
- 4 Theorems of Weierstrass and Montel . . . . . 290
- 5 Meromorphic functions . . . . . 290
  - 5.1 A complex Newton’s method . . . . . 291
  - 5.2 Sequences of complex numbers . . . . . 293

**2 Covering Spaces and the Monodromy Theorem**

**297**

- 1 Covering spaces and the lifting of curves . . . . . 297
  - 1.1 Examples of real or complex manifolds . . . . . 297
  - 1.2 Covering maps . . . . . 299
- 2 The sheaf of germs of holomorphic functions . . . . . 299
- 3 Covering spaces and integration along curves . . . . . 300
- 4 The monodromy theorem and the homotopy form of Cauchy’s theorem . . . . . 303
- 5 Applications of the monodromy theorem . . . . . 303

<b>3</b>	<b>The Winding Number and the Residue Theorem</b>	<b>305</b>
1	The winding number . . . . .	305
2	The residue theorem . . . . .	307
3	Applications of the residue theorem . . . . .	310
<b>4</b>	<b>Picard's Theorem</b>	<b>313</b>
<b>5</b>	<b>The Inhomogeneous Cauchy–Riemann Equation and Runge's Theorem</b>	<b>315</b>
1	Partitions of unity . . . . .	315
2	The equation $\partial u/\partial \bar{z} = \phi$ . . . . .	316
2.1	Complex differential forms . . . . .	316
2.2	Rouché's theorem . . . . .	321
2.3	Inhomogeneous Cauchy–Riemann equations . . . . .	322
3	Runge's theorem . . . . .	323
<b>6</b>	<b>Applications of Runge's Theorem</b>	<b>331</b>
1	The Mittag-Leffler theorem . . . . .	331
2	The cohomology form of Cauchy's theorem . . . . .	332
3	The theorem of Weierstrass . . . . .	332
4	Ideals in $\mathcal{H}(\Omega)$ . . . . .	335
<b>7</b>	<b>The Riemann Mapping Theorem and Simple Connectedness in the Plane</b>	<b>337</b>
1	Analytic automorphisms of the disc and of the annulus . . . . .	337
2	The Riemann mapping theorem . . . . .	340
3	Simply connected plane domains . . . . .	342
<b>8</b>	<b>Functions of Several Complex Variables</b>	<b>343</b>
<b>9</b>	<b>Compact Riemann Surfaces</b>	<b>351</b>
1	Definitions and basic theorems . . . . .	351
3	The cohomology group $H^1(\mathcal{U}, \mathcal{O})$ . . . . .	355
6	Meromorphic functions on a compact Riemann surface . . . . .	358
<b>10</b>	<b>The Corona Theorem</b>	<b>361</b>
1	The Poisson integral and the theorem of F. and M. Riesz . . . . .	361
<b>11</b>	<b>Subharmonic Functions and the Dirichlet Problem</b>	<b>365</b>
	<b>Notes for the exercises</b>	<b>369</b>
	<b>References for the exercises</b>	<b>373</b>
	<b>Index</b>	<b>379</b>