J.D. Murray

Mathematical Biology

II: Spatial Models and Biomedical Applications

Third Edition

With 298 Illustrations

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J.D. Murray, FRS
Emeritus Professor
University of Oxford *and*
University of Washington
Box 352420
Department of Applied Mathematics
Seattle, WA 98195-2420
USA

Editors

S.S. Antman
Department of Mathematics
and Institute for Physical Science
and Technology
University of Maryland
College Park, MD 20742-4015
USA
ssa@math.umd.edu

J.E. Marsden
Control and Dynamical Systems
Mail Code 107-81
California Institute of Technology
Pasadena, CA 91125
USA
marsden@cds.caltech.edu

L. Sirovich
Division of Applied Mathematics
Brown University
Providence, RI 02912
USA
chico@camelot.mssm.edu

S. Wiggins
School of Mathematics
University of Bristol
Bristol BS8 1TW
UK
s.wiggins@bris.ac.uk

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