

# **Applied Mathematical Sciences**

## **Volume 48**

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J. D. Murray

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With 25 Illustrations



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# Preface to the Springer Edition

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A further practical Chapter 7 has been added on matched asymptotic methods in singular perturbation theory and on multi-scale perturbation methods and suppression of secular terms.

## Preface

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‘A la verité, et ne crains point de l’aduouer, je porterois facilement, au besoing, une chandelle à Saint Michel, l’autre à son serpent.’†

MICHEL DE MONTAIGNE, 1533–1592

THIS book gives an introduction to the most frequently used methods for obtaining analytical approximations to functions defined by integrals or as solutions of ordinary differential equations. The emphasis throughout is on the practical use of the various techniques discussed. Heuristic reasoning, rather than mathematical rigor, is often used to justify a procedure, or some extension of it. This book is mainly intended for mathematicians and scientists whose primary aim is to get answers to practical problems. Frequently rigorous mathematical procedures are not available to deal with many of the problems which arise in practice and one of the aims of this book is to encourage the use of heuristic reasoning to get the solutions. Very often a rigorous development is suggested by a physical one. The philosophy behind the book is that, when solutions to non-standard problems are required, no procedure, be it rigorous or heuristic, should be scorned—a sentiment succinctly summed up by Montaigne.

The first chapter gives an introduction to asymptotic expansions while the next three present the main techniques, with many illustrative examples, for obtaining analytical approximations to integrals.

† This is from Montaigne’s essay *De l’utile et de l’honnête* (*Of usefulness and honesty*) and may be freely translated as ‘In truth, and I am not afraid to admit it, I would, in need, light a candle to Saint Michael and another to his dragon’.

Chapter 5 deals with the important class of integrals which arise from Fourier and Laplace transform solutions of differential equations. Chapter 6, the last and largest chapter, is concerned with asymptotic methods for ordinary differential equations. This subject is large and very much in current vogue under the title of singular perturbation theory. Here we give some practical methods which are in the nature of an introduction to the subject but which have a surprisingly wide applicability. In the bibliography and throughout the text various books for further reading and reference are listed.

This book is based on lectures given in the mathematics departments at Oxford University and New York University. The material can easily be covered in a single quarter term by final-year undergraduates or first-year graduates with some knowledge of functions of a complex variable and, for the last chapter, of ordinary differential equations.

I would like to thank Dr. A. B. Tayler and Mr. Peter Mitchell for the many helpful suggestions they made during the writing of this book. Finally I would like to express my special indebtedness to Dr. J. R. Ockendon for the many discussions I had with him and for his careful reading of earlier drafts of the text.

*Oxford*, 1973

J. D. M.

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