

# Fundamentals of Algorithms

FAIR

# Flexible Algorithms for Image Registration

# Jan Modersitzki

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