

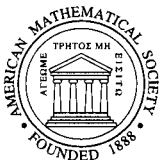
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**Calculus of Variations  
and Optimal Control**

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