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Comprehensive Mathematics for Computer Scientists 2

Calculus and ODEs, Splines, Probability, Fourier and Wavelet Theory, Fractals and Neural Networks, Categories and Lambda Calculus

With 114 Figures

Springer

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PART III

Topology and Calculus