

# **Applied Mathematical Sciences**

Volume 74

## *Editors*

F. John J.E. Marsden L. Sirovich

## *Advisors*

M. Ghil J.K. Hale J. Keller  
K. Kirchgässner B. Matkowsky  
J.T. Stuart A. Weinstein

# Applied Mathematical Sciences

---

1. *John*: Partial Differential Equations, 4th ed.
2. *Sirovich*: Techniques of Asymptotic Analysis.
3. *Hale*: Theory of Functional Differential Equations, 2nd ed.
4. *Percus*: Combinatorial Methods.
5. *von Mises/Friedrichs*: Fluid Dynamics.
6. *Freiberger/Grenander*: A Short Course in Computational Probability and Statistics.
7. *Pipkin*: Lectures on Viscoelasticity Theory.
9. *Friedrichs*: Spectral Theory of Operators in Hilbert Space.
11. *Wolovich*: Linear Multivariable Systems.
12. *Berkovitz*: Optimal Control Theory.
13. *Bluman/Cole*: Similarity Methods for Differential Equations.
14. *Yoshizawa*: Stability Theory and the Existence of Periodic Solution and Almost Periodic Solutions.
15. *Braun*: Differential Equations and Their Applications, 3rd ed.
16. *Lefschetz*: Applications of Algebraic Topology.
17. *Collatz/Wetterling*: Optimization Problems.
18. *Grenander*: Pattern Synthesis: Lectures in Pattern Theory, Vol I.
20. *Driver*: Ordinary and Delay Differential Equations.
21. *Courant/Friedrichs*: Supersonic Flow and Shock Waves.
22. *Rouche/Habets/Laloy*: Stability Theory by Liapunov's Direct Method.
23. *Lamperti*: Stochastic Processes: A Survey of the Mathematical Theory.
24. *Grenander*: Pattern Analysis: Lectures in Pattern Theory, Vol. II.
25. *Davies*: Integral Transforms and Their Applications, 2nd ed.
26. *Kushner/Clark*: Stochastic Approximation Methods for Constrained and Unconstrained Systems
27. *de Boor*: A Practical Guide to Splines.
28. *Keilson*: Markov Chain Models—Rarity and Exponentiality.
29. *de Veubeke*: A Course in Elasticity.
30. *Sniatycki*: Geometric Quantization and Quantum Mechanics.
31. *Reid*: Sturmian Theory for Ordinary Differential Equations.
32. *Meis/Markowitz*: Numerical Solution of Partial Differential Equations.
33. *Grenander*: Regular Structures: Lectures in Pattern Theory, Vol. III.
34. *Kevorkian/Cole*: Perturbation methods in Applied Mathematics.
35. *Carr*: Applications of Centre Manifold Theory.
36. *Bengtsson/Ghil/Källén*: Dynamic Meteorology: Data Assimilation Methods.
37. *Saperstone*: Semidynamical Systems in Infinite Dimensional Spaces.
38. *Lichtenberg/Lieberman*: Regular and Stochastic Motion.
39. *Piccini/Stampacchia/Vidossich*: Ordinary Differential Equations in  $\mathbb{R}^n$ .
40. *Naylor/Sell*: Linear Operator Theory in Engineering and Science.
41. *Sparrow*: The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors.
42. *Guckenheimer/Holmes*: Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields.
43. *Ockendon/Taylor*: Inviscid Fluid Flows.
44. *Pazy*: Semigroups of Linear Operators and Applications to Partial Differential Equations.
45. *Glashoff/Gustafson*: Linear Optimization and Approximation: An Introduction to the Theoretical Analysis and Numerical Treatment of Semi-Infinite Programs.
46. *Wilcox*: Scattering Theory for Diffraction Gratings.
47. *Hale et al.*: An Introduction to Infinite Dimensional Dynamical Systems—Geometric Theory.
48. *Murray*: Asymptotic Analysis.
49. *Ladyzhenskaya*: The Boundary-Value Problems of Mathematical Physics.
50. *Wilcox*: Sound Propagation in Stratified Fluids.
51. *Golubitsky/Schaeffer*: Bifurcation and Groups in Bifurcation Theory, Vol. I.
52. *Chipot*: Variational Inequalities and Flow in Porous Media.
53. *Majda*: Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables.
54. *Wasow*: Linear Turning Point Theory.

Jean Mawhin   Michel Willem

# Critical Point Theory and Hamiltonian Systems



Springer Science+Business Media, LLC

Jean Mawhin  
Institut de Mathematique Pure  
et Appliquee  
Chemin du Cyclotron 2  
1348 Louvain-la-Neuve  
Belgium

Michel Willem  
Institut de Mathematique Pure  
et Appliquee  
Chemin du Cyclotron 2  
1348 Louvain-la-Neuve  
Belgium

With 1 Illustration

---

Mathematics Subject Classification (1980): 58F05, 58E05, 70H25, 58F22, 58E07, 49A27, 58E30, 58-02, 49-02

---

Library of Congress Cataloging-in-Publication Data

Mawhin, J.

Critical point theory and Hamiltonian systems / Jean Mawhin,  
Michel Willem.

p. cm. — (Applied mathematical sciences ; v. 74)

Bibliography: p.

1. Hamiltonian systems. 2. Critical point theory (Mathematical analysis) I. Willem, Michel. II. Title. III. Series: Applied mathematical sciences (Springer-Verlag New York Inc.) ; v. 74.

QA1.A647 vol. 74

[QA614.83]

510 s—dc19

[515.3'5]

88-39058

Printed on acid-free paper.

ISBN 978-1-4419-3089-7 ISBN 978-1-4757-2061-7 (eBook)

DOI 10.1007/978-1-4757-2061-7

© 1989 by Springer Science+Business Media New York

Originally published by Springer-Verlag New York Inc in 1989.

Softcover reprint of the hardcover 1st edition 1989

All rights reserved. This work may not be translated or copied in whole or in part without the written permission of the publisher ( Springer Science+Business Media, LLC

), except for brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed is forbidden.

The use of general descriptive names, trade names, trademarks, etc. in this publication, even if the former are not especially identified, is not to be taken as a sign that such names, as understood by the Trade Marks and Merchandise Marks Act, may accordingly be used freely by anyone.

Camera-ready copy prepared using LaTeX.

Printed and bound by R.R. Donnelley & Sons, Harrisonburg, Virginia.

9 8 7 6 5 4 3 2 1

*To Margaret, Marie, Valérie, Jean  
and  
Eliane, Julie, Olivier*

# Preface

The formulation of laws of nature in terms of minimum principles has a long history that can be traced to Hero of Alexandria (c. 125 B.C.). He proved in his *Catoptrics* that when a ray of light is reflected by a mirror, the path actually taken from the object to the observer's eye is shorter than any other possible path so reflected. This principle was generalized by Fermat who postulated, around 1650, that light always propagates in the shortest time from one point to another, and deduced mathematically, from this principle, the law of refraction. The same Fermat anticipated the differential calculus by stating a necessary condition for the maximum or the minimum of a polynomial that is equivalent to the vanishing of its derivative.

More ambitious was the aim of Maupertuis when he enunciated, around 1750, his principle of least action as a rational and metaphysical basis for geometrical objects and mechanics. His statement was far from precise and, in the same year, Euler expressed it as an exact theorem of dynamics in an addendum to his famous book on the calculus of variations. This book contains the famous extension of the Fermat necessary condition for an extremum of a real function to the case of functionals of the type

$$y \rightarrow \int_a^b f(x, y(x), y'(x)) dx,$$

called the Euler–Lagrange equation after the more analytical treatment given shortly after by Lagrange.

It will take some time, during which further necessary conditions for a maximum or a minimum will be derived by Legendre, Jacobi, Weierstrass, and others to realize with Volterra and Hadamard, at the turn of this century, that the calculus of variations is just a special chapter of a theory of extrema for real functions defined on function spaces, and to create the tools necessary to formulate, in this setting, the corresponding necessary conditions.

The question of the existence of an extremum has a more recent history, a feature shared with the more general problem of existence theorems in mathematics. Gauss, who gave four demonstrations of the fundamental theorem of algebra, admitted without proof the existence of a minimum

for the functional  $\varphi$  given by

$$\varphi(y) = \int_{\Omega} \sum_{i=1}^n (D_i y(x))^2 dx$$

over all sufficiently regular functions  $y$  whose restriction on the boundary  $\partial\Omega$  of the bounded domain  $\Omega \subset \mathbf{R}^n$  is fixed. It was the origin of the so-called Dirichlet principle for the existence of a solution to the Dirichlet problem with data  $h$  on  $\partial\Omega$ ,

$$\begin{aligned} \Delta y(x) &= 0, & x \in \Omega \\ y(x) &= h(x), & x \in \partial\Omega. \end{aligned}$$

The long-awaited justification of this principle by Arzela and Hilbert, around 1900, was the stimulus for the creation of a systematic approach for getting conditions of existence for a minimum or a maximum of a functional.

After some pioneering work of Lebesgue, it became clear with Tonelli's important contributions that the lower semi-continuity introduced by Baire in another context was the right type of continuity for a fruitful abstract formulation of the calculus of variations. The systematic development of functional analysis, and in particular the study of convex sets and reflexive Banach spaces, paved the way for a systematic development of sharp existence conditions.

The creation of a general theory of periodic solutions of Hamiltonian systems as a fundamental step in understanding the structure of their solution set was one of the major motivations of Poincaré's monumental mathematical work. Besides many other contributions, Poincaré initiated the variational treatment of those questions. In particular, he made use of Jacobi's form of the least action principle to study the closed orbits of a conservative system with two degrees of freedom. He also considered the related question of the existence of geodesics. However, despite the rigorous treatment of the closed orbits of dynamical systems with two degrees of freedom by Whittaker, and the related work of Signorini, Tonelli, and Birkhoff, and despite the fact that Birkhoff minimax theory was the impetus for Morse theory and Lusternik-Schnirelman approach to critical point theory, progress toward a global variational approach for the periodic solutions of Hamiltonian systems was very slow.

A notable exception was Seifert's use in 1948 of Jacobi's form of the least action principle and differential geometry to prove the existence of an even  $T$ -periodic solution when the Hamiltonian is the sum of a kinetic and a potential energy term. This was generalized by Weinstein in the late 70s, who proved in particular, by similar methods, that an autonomous system with Hamiltonian  $H$  such that  $H^{-1}(1)$  is a manifold bounding a compact convex region always has a closed orbit in  $H^{-1}(1)$ .

The fundamental difficulty in applying the naive idea of finding the periodic solutions of a general Hamiltonian system through the critical points

of its Hamiltonian action on a suitable space of periodic functions lies in the fact, already observed by Birkhoff, that this action is unbounded from below and from above. This makes the use of the well-developed direct method of the calculus of variations (which deals with absolute minima) unapplicable, except in some particular second order systems already considered in the 20s by Lichtenstein and Hamel.

However, in the mid 60s, extensions of the minimax approach (in particular of Lusternik–Schnirelman theory) and of the Morse theory to functions defined on Banach manifolds were given by Palais, Smale, Rothe, Clark, Ambrosetti, Rabinowitz, and others. In the late 70s, Rabinowitz initiated the use of those methods in the study of periodic solutions of Hamiltonian systems. Later, a dual least action principle was introduced by Clarke and extensively developed by Clarke, Ekeland, and others. More recently, Morse theory and an extension of it due to Conley have provided further insight into those questions.

The aim of this book is to initiate the reader to those fundamental techniques of critical point theory and apply them to periodic solutions problems for Hamiltonian systems. Those illustrations have been chosen either because of their importance in the various applications in mechanics, electronics, and economics, or because of their mathematical importance. We hope that our style of presentation will be appealing to people trained and interested in ordinary differential equations. We have the feeling that critical point theory, which has been mostly developed by specialists in differential topology, partial differential equations, or optimization, should be made more popular among people working in ordinary differential equations. Of course, the variational methods developed here are directly applicable to partial differential equations problems at the expense of a substantial complication of the technical details. They can be found in a number of the references to the literature at the end of the book.

The reader interested in other aspects of critical point theory can then consult the references given in the bibliographical notes ending each chapter as well as the following surveys and monographs: [AuE<sub>1</sub>], [Berc<sub>1</sub>], [Ber<sub>2</sub>], [Blo<sub>1</sub>], [Bot<sub>1,2</sub>], [Bre<sub>2</sub>], [Ces<sub>1</sub>], [Cha<sub>1</sub>], [ChH<sub>1</sub>], [Cla<sub>3</sub>], [Con<sub>1</sub>], [Cor<sub>1</sub>], [Dei<sub>1</sub>], [Des<sub>1</sub>], [Eel<sub>1</sub>], [Eke<sub>5</sub>], [EkT<sub>1</sub>], [EkTu<sub>1</sub>], [Fen<sub>1</sub>], [Fun<sub>1</sub>], [Kli<sub>1</sub>], [Koz<sub>1</sub>], [Kra<sub>2</sub>], [Lju<sub>1</sub>], [Maw<sub>2,3,5,11</sub>], [Mil<sub>2</sub>], [Mor<sub>3</sub>], [Moy<sub>1</sub>], [Mrs<sub>2</sub>], [Nir<sub>2</sub>], [Rab<sub>2,6,12,13,14,19</sub>], [Roc<sub>1</sub>], [Rot<sub>5</sub>], [Ryb<sub>1</sub>], [Sch<sub>1</sub>], [Smo<sub>1</sub>], [Str<sub>4,5</sub>], [Szu<sub>3</sub>], [Ton<sub>1</sub>], [Vai<sub>1,2</sub>], [Vol<sub>1,2</sub>], [VoP<sub>1</sub>], [Wil<sub>3,5</sub>], [You<sub>2</sub>], [Zeh<sub>1</sub>], [Zei<sub>2</sub>].

## Acknowledgments

We wish to thank the people who have contributed to the realization of this work.

The manuscript benefited from suggestions and criticisms by our colleagues C. Fabry and P. Habets, and our students A. Fonda, C. Gorez, W. Omana, and S. Tshinanga, who have read parts of the manuscript.



We are indebted to the editorial board of Springer-Verlag for carefully reviewing the manuscript.

We are very grateful to Béatrice Huberty for her accurate and superb typing of the manuscript.

Finally we wish to thank our families for their patience during the elaboration of this book.

Louvain-la-Neuve

Jean Mawhin  
Michel Willem

# Contents

Preface		vii
1	The Direct Method of the Calculus of Variations	1
	Introduction	1
1.1	Lower Semi-Continuous Functions	3
1.2	Convex Functions	4
1.3	Euler Equation	5
1.4	The Calculus of Variations with Periodic Boundary Conditions	6
1.5	Periodic Solutions of Non-Autonomous Second Order Systems with Bounded Nonlinearity	12
1.6	Periodic Solutions of Non-Autonomous Second Order Systems with Periodic Potential	14
1.7	Periodic Solutions of Non-Autonomous Second Order Systems with Convex Potential	17
	Historical and Bibliographical Notes	22
	Exercises	25
2	The Fenchel Transform and Duality	28
	Introduction	28
2.1	Definition of the Fenchel Transform	30
2.2	Differentiable Convex Functions	34
2.3	Hamiltonian Duality	35
2.4	Clarke Duality	37
	Historical and Bibliographical Notes	39
	Exercises	40
3	Minimization of the Dual Action	42
	Introduction	42
3.1	Eigenvalues of Eigenfunctions of $J(d/dt)$ with Periodic Boundary Conditions	43

3.2	A Basic Existence Theorem for Periodic Solutions of Convex Hamiltonian Systems	45
3.3	Subharmonics of Non-Autonomous Convex Hamiltonian Systems	50
3.4	Periodic Solutions with Prescribed Minimal Period of Autonomous Convex Hamiltonian Systems	53
3.5	Periodic Solutions with Prescribed Energy of Autonomous Hamiltonian Systems	56
3.6	Periodic Solutions of Non-Autonomous Second Order Systems with Convex Potential	60
3.7	A Variant of the Dual Least Action Principle for Non-Autonomous Second Order Systems	61
3.8	The Range of Some Second Order Nonlinear Operators with Periodic Boundary Conditions	67
	Historical and Bibliographical Notes	70
	Exercises	71
4	Minimax Theorems for Indefinite Functionals	73
	Introduction	73
4.1	Ekeland's Variational Principle and the Existence of Almost Critical Points	75
4.2	A Closedness Condition and the Existence of Critical Points	81
4.3	The Saddle Point Theorem and Periodic Solutions of Second Order Systems with Bounded Nonlinearity	85
4.4	Periodic Solutions of Josephson-Type Systems	87
4.5	The Mountain Pass Theorem and Periodic Solutions of Superlinear Convex Autonomous Hamiltonian Systems	92
4.6	Multiple Critical Points of Periodic Functionals	97
	Historical and Bibliographical Notes	106
	Exercises	107
5	A Borsuk–Ulam Theorem and Index Theories	111
	Introduction	111
5.1	Group Representations	112
5.2	The Parametrized Sard Theorem	114
5.3	Topological Degree	116
5.4	Index Theories	120
	Historical and Bibliographical Notes	123
	Exercises	124

6	Lusternik–Schnirelman Theory and Multiple Periodic Solutions with Fixed Energy	126
	Introduction	126
6.1	Equivariant Deformations	127
6.2	Existence of Multiple Critical Points	132
6.3	Multiple Periodic Solutions with Prescribed Energy of Autonomous Hamiltonian Systems	134
6.4	Nonlinear Eigenvalue Problems	139
6.5	Application to Bifurcation Theory	141
6.6	Multiple Periodic Solutions with Prescribed Energy Near an Equilibrium	148
	Historical and Bibliographical Notes	149
	Exercises	151
7	Morse–Ekeland Index and Multiple Periodic Solutions with Fixed Period	153
	Introduction	153
7.1	The Index of a Linear Positive Definite Hamiltonian System	154
7.2	Linear Autonomous Positive Definite Hamiltonian Systems	160
7.3	Periodic Solutions of Convex Asymptotically Linear Autonomous Hamiltonian Systems	161
	Historical and Bibliographical Notes	165
	Exercises	165
8	Morse Theory	167
	Introduction	167
8.1	Relative Homology	168
8.2	Manifolds	173
8.3	Vector Fields	175
8.4	Riemannian Manifolds	176
8.5	Morse Inequalities	179
8.6	The Generalized Morse Lemma	184
8.7	Computation of the Critical Groups	189
8.8	Critical Groups at a Point of Mountain Pass Type	195
8.9	Continuity of the Critical Groups and Bifurcation Theory	196
8.10	Lower Semi-Continuity of the Betti Numbers	199
8.11	Critical Groups at a Saddle Point	200

	Historical and Bibliographical Notes	201
	Exercises	203
9	Applications of Morse Theory to Second Order Systems	205
	Introduction	205
9.1	The Index of a Linear Second Order Differential System	206
9.2	Periodic Solutions of Autonomous Second Order Systems Near an Equilibrium	207
9.3	Periodic Solutions of Asymptotically Linear Non-Autonomous Second Order Systems	210
9.4	Multiple Solutions of Lagrangian Systems	214
	Historical and Bibliographical Notes	215
	Exercises	216
10	Nondegenerate Critical Manifolds	217
	Introduction	217
10.1	Submanifolds	217
10.2	Normal Bundle	219
10.3	Critical Groups of a Nondegenerate Critical Manifold	221
10.4	Global Theory	224
10.5	Second Order Autonomous Superlinear Equations	227
10.6	Periodic Solutions of Forced Superlinear Second Order Equations	231
10.7	Local Perturbations of Nondegenerate Critical Manifolds	235
	Historical and Bibliographical Notes	238
	Exercises	238
	Bibliography	240
	Index	275