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Critical Point Theory and Hamiltonian Systems

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To Margaret, Marie, Valérie, Jean
and
Eliane, Julie, Olivier

Preface

The formulation of laws of nature in terms of minimum principles has a long history that can be traced to Hero of Alexandria (c. 125 B.C.). He proved in his Catoptrics that when a ray of light is reflected by a mirror, the path actually taken from the object to the observer's eye is shorter than any other possible path so reflected. This principle was generalized by Fermat who postulated, around 1650, that light always propagates in the shortest time from one point to another, and deduced mathematically, from this principle, the law of refraction. The same Fermat anticipated the differential calculus by stating a necessary condition for the maximum or the minimum of a polynomial that is equivalent to the vanishing of its derivative.

More ambitious was the aim of Maupertuis when he enunciated, around 1750, his principle of least action as a rational and metaphysical basis for geometrical objects and mechanics. His statement was far from precise and, in the same year, Euler expressed it as an exact theorem of dynamics in an addendum to his famous book on the calculus of variations. This book contains the famous extension of the Fermat necessary condition for an extremum of a real function to the case of functionals of the type

$$y \to \int_a^b f(x, y(x), y'(x))\, dx,$$

called the Euler–Lagrange equation after the more analytical treatment given shortly after by Lagrange.

It will take some time, during which further necessary conditions for a maximum or a minimum will be derived by Legendre, Jacobi, Weierstrass, and others to realize with Volterra and Hadamard, at the turn of this century, that the calculus of variations is just a special chapter of a theory of extrema for real functions defined on function spaces, and to create the tools necessary to formulate, in this setting, the corresponding necessary conditions.

The question of the existence of an extremum has a more recent history, a feature shared with the more general problem of existence theorems in mathematics. Gauss, who gave four demonstrations of the fundamental theorem of algebra, admitted without proof the existence of a minimum for the functional φ given by

$$\varphi(y) = \int_{\Omega} \sum_{i=1}^{n} (D_i y(x))^2 \, dx$$

over all sufficiently regular functions y whose restriction on the boundary $\partial\Omega$ of the bounded domain $\Omega \subset \mathbf{R}^n$ is fixed. It was the origin of the so-called Dirichlet principle for the existence of a solution to the Dirichlet problem with data h on $\partial\Omega$,

$$\Delta y(x) = 0, \quad x \in \Omega$$
$$y(x) = h(x), \quad x \in \partial\Omega.$$

The long-waited justification of this principle by Arzela and Hilbert, around 1900, was the stimulus for the creation of a systematic approach for getting conditions of existence for a minimum or a maximum of a functional.

After some pioneering work of Lebesgue, it became clear with Tonelli's important contributions that the lower semi-continuity introduced by Baire in another context was the right type of continuity for a fruitful abstract formulation of the calculus of variations. The systematic development of functional analysis, and in particular the study of convex sets and reflexive Banach spaces, paved the way for a systematic development of sharp existence conditions.

The creation of a general theory of periodic solutions of Hamiltonian systems as a fundamental step in understanding the structure of their solution set was one of the major motivations of Poincaré's monumental mathematical work. Besides many other contributions, Poincaré initiated the variational treatment of those questions. In particular, he made use of Jacobi's form of the least action principle to study the closed orbits of a conservative system with two degrees of freedom. He also considered the related question of the existence of geodesics. However, despite the rigorous treatment of the closed orbits of dynamical systems with two degrees of freedom by Whittaker, and the related work of Signorini, Tonelli, and Birkhoff, and despite the fact that Birkhoff minimax theory was the impetus for Morse theory and Lusternik–Schnirelman approach to critical point theory, progress toward a global variational approach for the periodic solutions of Hamiltonian systems was very slow.

A notable exception was Seifert's use in 1948 of Jacobi's form of the least action principle and differential geometry to prove the existence of an even T-periodic solution when the Hamiltonian is the sum of a kinetic and a potential energy term. This was generalized by Weinstein in the late 70s, who proved in particular, by similar methods, that an autonomous system with Hamiltonian H such that $H^{-1}(1)$ is a manifold bounding a compact convex region always has a closed orbit in $H^{-1}(1)$.

The fundamental difficulty in applying the naive idea of finding the periodic solutions of a general Hamiltonian system through the critical points of its Hamiltonian action on a suitable space of periodic functions lies in the fact, already observed by Birkhoff, that this action is unbounded from below and from above. This makes the use of the well-developed direct method of the calculus of variations (which deals with absolute minima) unapplicable, except in some particular second order systems already considered in the 20s by Lichtenstein and Hamel.

However, in the mid 60s, extensions of the minimax approach (in particular of Lusternik–Schnirelman theory) and of the Morse theory to functions defined on Banach manifolds were given by Palais, Smale, Rothe, Clark, Ambrosetti, Rabinowitz, and others. In the late 70s, Rabinowitz initiated the use of those methods in the study of periodic solutions of Hamiltonian systems. Later, a dual least action principle was introduced by Clarke and extensively developed by Clarke, Ekeland, and others. More recently, Morse theory and an extension of it due to Conley have provided further insight into those questions.

The aim of this book is to initiate the reader to those fundamental techniques of critical point theory and apply them to periodic solutions problems for Hamiltonian systems. Those illustrations have been chosen either because of their importance in the various applications in mechanics, electronics, and economics, or because of their mathematical importance. We hope that our style of presentation will be appealing to people trained and interested in ordinary differential equations. We have the feeling that critical point theory, which has been mostly developed by specialists in differential topology, partial differential equations, or optimization, should be made more popular among people working in ordinary differential equations. Of course, the variational methods developed here are directly applicable to partial differential equations problems at the expense of a substantial complication of the technical details. They can be found in a number of the references to the literature at the end of the book.

The reader interested in other aspects of critical point theory can then consult the references given in the bibliographical notes ending each chapter as well as the following surveys and monographs: [AuE$_1$], [Berc$_1$], [Ber$_2$], [Blo$_1$], [Bot$_{1,2}$], [Bre$_2$], [Ces$_1$], [Cha$_1$], [ChH$_1$], [Cla$_3$], [Con$_1$], [Cor$_1$], [Dei$_1$], [Des$_1$], [Eel$_1$], [Eke$_5$], [EkT$_1$], [EkTu$_1$], [Fen$_1$], [Fun$_1$], [Kli$_1$], [Koz$_1$], [Kra$_2$], [Lju$_1$], [Maw$_{2,3,5,11}$], [Mil$_2$], [Mor$_3$], [Moy$_1$], [Mrs$_2$], [Nir$_2$], [Rab$_{2,6,12,13,14,19}$], [Roc$_1$], [Rot$_5$], [Ryb$_1$], [Sch$_1$], [Smo$_1$], [Str$_{4,5}$] [Szu$_3$], [Ton$_1$], [Vai$_{1,2}$], [Vol$_{1,2}$], [VoP$_1$], [Wil$_{3,5}$], [You$_2$], [Zeh$_1$], [Zei$_2$].

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Louvain-la-Neuve

Jean Mawhin
Michel Willem

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