Jerrold E. Marsden Tudor S. Ratiu

Introduction to Mechanics and Symmetry

A Basic Exposition of Classical Mechanical Systems

Second Edition

With 54 Illustrations

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California Institute of Technology
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USA

Tudor S. Ratiu
Ecole polytechnique fédérale
de Lausanne
Département de mathématiques
CH-1015 Lausanne
Switzerland

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