

Undergraduate Texts in Mathematics

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Introduction to Mathematical Logic

Set Theory
Computable Functions
Model Theory



Springer-Verlag
New York Heidelberg Berlin

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AMS Subject Classification: 02-01, 04-01

With 2 Figures

Library of Congress Cataloging in Publication Data

Malitz, J.
Introduction to mathematical logic.

Bibliography: p.
Includes index.

1. Logic, Symbolic and mathematical. I. Title.
QA9.M265 511'.3 78-13588

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© 1979 by Springer-Verlag New York Inc.
Softcover reprint of the hardcover 1st edition 1979

9 8 7 6 5 4 3 2 1

ISBN-13 : 978-1-4613-9443-3
DOI : 10.1007 / 978-1-4613-9441-9

e-ISBN-13 : 978-1-4613-9441-9

For Sue, Jed, and Seth

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Preface

This book is intended as an undergraduate senior level or beginning graduate level text for mathematical logic. There are virtually no prerequisites, although a familiarity with notions encountered in a beginning course in abstract algebra such as groups, rings, and fields will be useful in providing some motivation for the topics in Part III.

An attempt has been made to develop the beginning of each part slowly and then to gradually quicken the pace and the complexity of the material. Each part ends with a brief introduction to selected topics of current interest.

The text is divided into three parts: one dealing with set theory, another with computable function theory, and the last with model theory. Part III relies heavily on the notation, concepts and results discussed in Part I and to some extent on Part II. Parts I and II are independent of each other, and each provides enough material for a one semester course.

The exercises cover a wide range of difficulty with an emphasis on more routine problems in the earlier sections of each part in order to familiarize the reader with the new notions and methods. The more difficult exercises are accompanied by hints. In some cases significant theorems are developed step by step with hints in the problems. Such theorems are not used later in the sequence.

The part dealing with set theory is intended to provide a notational and conceptual framework for areas of mathematics outside of logic as well as to introduce the student to those topics that are of particular interest to those working in the foundations of set theory.

We hope that the part of the text devoted to computable functions will be of interest to those who intend to work with real world computers.

We believe that the notation, methodology, and results of elementary logic should be a part of a general mathematics program and are of value in a wide variety of disciplines within mathematics and outside of mathematics.

Boulder, Colorado
March 1979

J. MALITZ

Glossary of Symbols

$\{\dots\}$	2	f^{-1}	7
I	2	$f \upharpoonright C$	7
N	2	$f \circ g$	7
\mathbb{N}^+	2	\sim	9
Q	2	$<, \leq$	15
\mathbb{Q}^+	2	$(\mathbb{R}, <), (\mathbb{Q}, <), (\mathbb{I}, <),$	22
R	2	$<_A$	
\mathbb{R}^+	2	$(\mathbb{N}, <)$	23
$\{x: \dots\}$	2	Ord α	33
\emptyset	2	\cong	34
\in	2	Card	36
$\subseteq, \supseteq, \subset, \supset$	2, 3	$c(x)$	37
\cup	3	ZF	38
$\cup X$	4	ZFC	46
$\cup_{i \in I} A_i$	4	\vDash	51
\cap	4	$M(t)$	61
$\cap X$	4	Sum	63
$B - A$	4	$C_{k,d}$	63
$P(X)$	5	$P_{k,t}$	63
$A \times B$	6	Pred	63
$[B]_k$	6	Prod $_n$	69
Dom R	7	Mult	70
Ranz R	7	Pow	71
$1-1$	7	Diff'	71
$f: A \rightarrow B$	7	$m \dot{-} n$	71
$\wedge B$	7	$\exists x < y$	75
$f[C]$	7	$\forall x < y$	75
		$P(\bar{n}, x)$	75
		Prime	75

Prim	75
Exp'	76
Max	77
$M t \rightsquigarrow s$	80
\dot{n}	80
compress	80
M_1 ↓ M	82
$M_1 \leftarrow M, M \rightarrow M_1$ M $M_1 \leftarrow M \rightarrow M_2$ $\curvearrowright M, M \curvearrowleft$	82
copy k	83
shift right	84
shift left	84
erase	84
# k	91
TS	95
STP	95
decode	95
code	98
Exp	98
RR	98
RC	98
NP	99
NS	99
NST	99
T	99
STP	99
Row	100
Mach	101

In	101
Halt	101
\forall	103
\exists	103
Rec	107
Rem	108
L	111
(see also 136)	
\approx	111
\vee, \wedge, \neg	111
\forall, \exists	111
\oplus, \odot	111
[,]	111
Trm	111
$t\langle z \rangle$	112
Fm	113
\vDash	114
\vdash	124
Cons _S	129
Prf _S	129
P	133
NP	133
τ	136
Fm _S	137
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$\mathfrak{B} \upharpoonright s$	140
\cong, \equiv	140
$z \binom{u}{a}$	142
$t^{\mathfrak{A}}\langle z \rangle$	142
$\mathfrak{A} \vDash \varphi\langle z \rangle$	142
Th \mathfrak{A}	144
\equiv	144
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α	167
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$\bigcup_{\alpha < k} \mathfrak{A}_\alpha$	170
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$\mathfrak{A} \alpha_T \mathfrak{B}$	178
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