

*Stochastic Mechanics*  
*Random Media*  
*Signal Processing*  
*and Image Synthesis*  
*Mathematical Economics*  
*Stochastic Optimization*  
*Stochastic Control*

**Applications of  
Mathematics**  
*Stochastic Modelling  
and Applied Probability*

**32**

*Edited by* I. Karatzas  
M. Yor

*Advisory Board* P. Brémaud  
E. Carlen  
R. Dobrushin  
W. Fleming  
D. Geman  
G. Grimmett  
G. Papanicolaou  
J. Scheinkman

**Springer**

*New York*  
*Berlin*  
*Heidelberg*  
*Barcelona*  
*Budapest*  
*Hong Kong*  
*London*  
*Milan*  
*Paris*  
*Santa Clara*  
*Singapore*  
*Tokyo*

# Applications of Mathematics

---

- 1 Fleming/Rishel, **Deterministic and Stochastic Optimal Control** (1975)
- 2 Marchuk, **Methods of Numerical Mathematics**, Second Ed. (1982)
- 3 Balakrishnan, **Applied Functional Analysis**, Second Ed. (1981)
- 4 Borovkov, **Stochastic Processes in Queueing Theory** (1976)
- 5 Liptser/Shiryayev, **Statistics of Random Processes I: General Theory** (1977)
- 6 Liptser/Shiryayev, **Statistics of Random Processes II: Applications** (1978)
- 7 Vorob'ev, **Game Theory: Lectures for Economists and Systems Scientists** (1977)
- 8 Shiryayev, **Optimal Stopping Rules** (1978)
- 9 Ibragimov/Rozanov, **Gaussian Random Processes** (1978)
- 10 Wonham, **Linear Multivariable Control: A Geometric Approach**, Third Ed. (1985)
- 11 Hida, **Brownian Motion** (1980)
- 12 Hestenes, **Conjugate Direction Methods in Optimization** (1980)
- 13 Kallianpur, **Stochastic Filtering Theory** (1980)
- 14 Krylov, **Controlled Diffusion Processes** (1980)
- 15 Prabhu, **Stochastic Storage Processes: Queues, Insurance Risk, and Dams** (1980)
- 16 Ibragimov/Has'minskii, **Statistical Estimation: Asymptotic Theory** (1981)
- 17 Cesari, **Optimization: Theory and Applications** (1982)
- 18 Elliott, **Stochastic Calculus and Applications** (1982)
- 19 Marchuk/Shaidourov, **Difference Methods and Their Extrapolations** (1983)
- 20 Hijab, **Stabilization of Control Systems** (1986)
- 21 Protter, **Stochastic Integration and Differential Equations** (1990)
- 22 Benveniste/Métivier/Priouret, **Adaptive Algorithms and Stochastic Approximations** (1990)
- 23 Kloeden/Platen, **Numerical Solution of Stochastic Differential Equations** (1992)
- 24 Kushner/Dupuis, **Numerical Methods for Stochastic Control Problems in Continuous Time** (1992)
- 25 Fleming/Soner, **Controlled Markov Processes and Viscosity Solutions** (1993)
- 26 Baccelli/Brémaud, **Elements of Queueing Theory** (1994)
- 27 Winkler, **Image Analysis, Random Fields, and Dynamic Monte Carlo Methods: An Introduction to Mathematical Aspects** (1994)
- 28 Kalpazidou, **Cycle Representations of Markov Processes** (1995)
- 29 Elliott/Aggoun/Moore, **Hidden Markov Models: Estimation and Control** (1995)
- 30 Hernández-Lerma/Lasserre, **Discrete-Time Markov Control Processes: Basic Optimality Criteria** (1996)
- 31 Devroye/Györfi/Lugosi, **A Probabilistic Theory of Pattern Recognition** (1996)
- 32 Maitra/Sudderth, **Discrete Gambling and Stochastic Games** (1996)

Ashok P. Maitra William D. Sudderth

# Discrete Gambling and Stochastic Games



Springer

Ashok P. Maitra  
William D. Sudderth  
College of Liberal Arts  
School of Statistics  
University of Minnesota  
Minneapolis, MN 55455-0488  
USA

*Managing Editors*

I. Karatzas  
Department of Statistics  
Columbia University  
New York, NY 10027, USA

M. Yor  
CNRS, Laboratoire de Probabilités  
Université Pierre et Marie Curie  
4, Place Jussieu, Tour 56  
F-75252 Paris Cedex 05, France

---

Mathematics Subject Classification (1991): Primary—60G40, 90D15; Secondary—28A12, 90C39, 90C40, 90D60

---

With 1 figure.

Library of Congress Cataloging-in-Publication Data

Maitra, Ashok P.

Discrete gambling and stochastic games/by Ashok P. Maitra and  
William D. Sudderth

p. cm. — (Applications of mathematics; 32)

Includes bibliographical references and index.

1. Games of chance (Mathematics). 2. Gambling. 3. Stochastic  
inequalities. I. Sudderth, William D. II. Title. III. Series.

QA271.M35 1996

795'.01'5193—dc20

95-44636

Printed on acid-free paper.

ISBN-13: 978-1-4612-8467-3 e-ISBN-13: 978-1-4612-4002-0

DOI: 10.1007/978-1-4612-4002-0

© 1996 Springer-Verlag New York, Inc.

Softcover reprint of the hardcover 1st edition 1996

All rights reserved. This work may not be translated or copied in whole or in part without the written permission of the publisher (Springer-Verlag New York, Inc., 175 Fifth Avenue, New York, NY 10010, USA), except for brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed is forbidden.

The use of general descriptive names, trade names, trademarks, etc., in this publication, even if the former are not especially identified, is not to be taken as a sign that such names, as understood by the Trade Marks and Merchandise Marks Act, may accordingly be used freely by anyone.

Production managed by Francine McNeill; manufacturing supervised by Jacqui Ashri.  
Photocomposed copy prepared using the authors' LaTeX files.

To Indrani and Jerri

# Acknowledgments

We are indebted to a number of friends for their suggestions and constructive criticism of drafts of this book. Especially valuable were the comments of Bert Fristedt, David Heath, Victor Pestien, S. Ramakrishnan, and Piercesare Secchi.

Roger Purves, who wisely declined to be a co-author, nevertheless helped us with many stylistic, as well as mathematical, points. Ioannis Karatzas suggested that we write the book and encouraged us to make it as accessible as possible.

Courses were taught from early drafts at the University of California at Berkeley and at the University of Minnesota. We appreciate having had the opportunity to teach this material and are grateful to the students who participated.

The influence of our teachers, David Blackwell and Lester Dubins, pervades the entire manuscript. We hope it is not too far below their high standards.

It was a pleasure to work with Melanie Anton, who typed and retyped the entire book. Her cheerful acceptance of our fussy requests made it easier for us to survive in an era when nontypists are an endangered species.

Sudderth's research for the book was partially supported by the National Science Foundation under Grants DMS-9123358 and DMS-9423009.

# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
1.1	Preview . . . . .	2
1.2	Prerequisites . . . . .	2
1.3	Numbering . . . . .	2
<b>2</b>	<b>Gambling Houses and the Conservation of Fairness</b>	<b>5</b>
2.1	Introduction . . . . .	5
2.2	Gambles, Gambling Houses, and Strategies . . . . .	6
2.3	Stopping Times and Stop Rules . . . . .	9
2.4	An Optional Sampling Theorem . . . . .	11
2.5	Martingale Convergence Theorems . . . . .	15
2.6	The Ordinals and Transfinite Induction . . . . .	15
2.7	Uncountable State Spaces and Continuous-Time . . . . .	18
2.8	Problems for Chapter 2 . . . . .	19
<b>3</b>	<b>Leavable Gambling Problems</b>	<b>23</b>
3.1	The Fundamental Theorem . . . . .	24
3.2	The One-Day Operator and the Optimality Equation . . . . .	26
3.3	The Utility of a Strategy . . . . .	27
3.4	Some Examples . . . . .	30
3.5	Optimal Strategies . . . . .	42
3.6	Backward Induction: An Algorithm for $U$ . . . . .	48
3.7	Problems for Chapter 3 . . . . .	52

<b>4</b>	<b>Nonleavable Gambling Problems</b>	<b>59</b>
4.1	Introduction . . . . .	59
4.2	Understanding $u(\sigma)$ . . . . .	60
4.3	A Characterization of $V$ . . . . .	68
4.4	The Optimality Equation for $V$ . . . . .	69
4.5	Proving Optimality . . . . .	70
4.6	Some Examples . . . . .	70
4.7	Optimal Strategies . . . . .	75
4.8	Another Characterization of $V$ . . . . .	78
4.9	An Algorithm for $V$ . . . . .	82
4.10	Problems for Chapter 4 . . . . .	84
<b>5</b>	<b>Stationary Families of Strategies</b>	<b>89</b>
5.1	Introduction . . . . .	89
5.2	Comparing Strategies . . . . .	90
5.3	Finite Gambling Problems . . . . .	94
5.4	Nonnegative Stop-or-Go Problems . . . . .	96
5.5	Leavable Houses . . . . .	101
5.6	An Example of Blackwell and Ramakrishnan . . . . .	106
5.7	Markov Families of Strategies . . . . .	109
5.8	Stationary Plans in Dynamic Programming . . . . .	109
5.9	Problems for Chapter 5 . . . . .	110
<b>6</b>	<b>Approximation Theorems</b>	<b>113</b>
6.1	Introduction . . . . .	113
6.2	Analytic Sets . . . . .	114
6.3	Optimality Equations . . . . .	124
6.4	Special Cases of Theorem 1.2 . . . . .	128
6.5	The Going-Up Property of $\overline{M}$ . . . . .	139
6.6	Dynamic Capacities and the Proof of Theorem 1.2 . . . . .	144
6.7	Approximating Functions . . . . .	150
6.8	Composition Closure and Saturated House . . . . .	158
6.9	Problems for Chapter 6 . . . . .	165
<b>7</b>	<b>Stochastic Games</b>	<b>171</b>
7.1	Introduction . . . . .	171
7.2	Two-Person, Zero-Sum Games . . . . .	172
7.3	The Dynamics of Stochastic Games . . . . .	176
7.4	Stochastic Games with $\limsup$ Payoff . . . . .	179
7.5	Other Payoff Functions . . . . .	180
7.6	The One-Day Operator . . . . .	181
7.7	Leavable Games . . . . .	184
7.8	Families of Optimal Strategies for Leavable Games . . . . .	189
7.9	Examples of Leavable Games . . . . .	191
7.10	A Modification of Leavable Games and the Operator $T$ . . . . .	196



7.11 An Algorithm for the Value of a Nonleavable Game . . . . .	198
7.12 The Optimality Equation for $V$ . . . . .	201
7.13 Good Strategies in Nonleavable Games . . . . .	203
7.14 Win, Lose, or Draw . . . . .	207
7.15 Recursive Matrix Games . . . . .	210
7.16 Games of Survival . . . . .	212
7.17 The Big Match . . . . .	216
7.18 Problems for Chapter 7 . . . . .	221
<b>References</b>	<b>227</b>
<b>Symbol Index</b>	<b>239</b>
<b>Index</b>	<b>241</b>