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978-1-107-02103-7 - Sets of Finite Perimeter and Geometric Variational Problems: An Introduction to Geometric Measure Theory

Francesco Maggi

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