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Dirichlet Integrals on Harmonic Spaces

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INTRODUCTION

The classical potential theory is, in a sense, a study of the Laplace equation $\Delta u=0$. It has been clarified that second order elliptic, and some parabolic, partial differential equations share many potential theoretic properties with the Laplace equation. An axiomatic potential theory tries to develop a unified method of treating these equations.

In an axiomatic potential theory, we start with defining a harmonic space $(X, \mathcal{H})$ or $(X, \mathcal{U})$, where X is locally compact Hausdorff space and $\mathcal{H}$ (resp. $\mathcal{U}$) is a sheaf of linear spaces of continuous functions (resp. convex cones of lower semicontinuous functions) which are called "harmonic" (resp. "hyperharmonic"). There are several different kinds of harmonic spaces so far introduced. Among them, the following three are the most well-established:

(a) Brelot's harmonic space $(X, \mathcal{H})$ (see [6], [7], [16], etc.);
(b) Harmonic spaces $(X, \mathcal{H})$ given in Bauer [1] and in Boboc-Constantinescu-Cornea [2];
(c) Harmonic space $(X, \mathcal{U})$ proposed in Constantinescu-Cornea [11].

On any of these harmonic spaces, we can naturally develop a theory of superharmonic functions and potentials, including the Perron-Wiener's method for Dirichlet problems, balayage theory and even integral representation of potentials; and thus a fairly large part of the classical potential theory is covered also by axiomatic theory.

There are, however, some important parts in the classical potential theory which involve the notion of Dirichlet integrals. Due to the fact that only topological notions and some order relations are involved in an axiomatic potential theory, it is impossible to define differentiation of functions without further structures on X. However, it appears that with some reasonable additional structure for $\mathcal{H}$ or $\mathcal{U}$, we can consider a notion corresponding to the gradient of functions on a harmonic space.

As an illustration, let us consider the case where X is an euclidean domain and the harmonic sheaf $\mathcal{H}$ is given by the solutions of the second order differential equation

$$Lu \equiv \Sigma\, a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \Sigma\, b_i \frac{\partial u}{\partial x_i} + cu = 0,$$

where a_{ij}, b_i, c are functions on X with certain regularity and (a_{ij}) is positive definite everywhere on X. Now, we have the following equality:

$$2\, \Sigma\, a_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} = L(fg) - fLg - gLf + fgL1.$$

This shows that the function $\Sigma\, a_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}$ (which, by an abuse of terminology, we call mutual gradient of f and g) can be expressed in terms of L. Therefore, in an axiomatic theory, once a notion correspon-ding to the operator L is introduced, then mutual gradients of functions can be defined by the above equation.

The purpose of the present lectures is to define the notion of (mutual) gradients of functions on harmonic spaces following the above idea, to show that this notion enjoys some basic properties possessed by the form $\Sigma\, a_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}$ and to develop some theories involving the notion of Dirichlet integrals in the axiomatic setting.

As a matter of fact, we define the gradients of functions as measures, which we call gradient measures. The definition and the verification of basic properties of gradient measures can be carried out on general harmonic spaces in the sense of Constantinescu-Cornea [11]. Thus, in Part I, we give a theory on general harmonic spaces. Sections §1 and §2 are preparatory and almost all materials in these sections are taken from Part I of [11]. In §3, we give the definition of gradient measures and prove basic properties. This section is nearly identical with [26].

In order to obtain richer results, it becomes necessary for us to restrict ourselves to self-adjoint harmonic spaces. Self-adjointness of a harmonic space is defined by the existence of consistent system of symmetric Green functions (see §4 for details); its prototype is the space given by solutions of the equation of the form $\Delta u = cu$ (c: a function). Thus, in Part II and Part III, we develop our theory on self-adjoint harmonic spaces. The main theme of Part II is Green's formula. In §4, we study Green potentials and in §5 we establish

Green's formula for a harmonic function and a potential both with finite energy. Most of the materials in Part II are taken from [24] (and also [22], [23]), but in these lectures arrangements and proofs are often different from those in [24] and the final form of Green's formula is improved. Part III is devoted to the study of various spaces of Dirichlet-finite or energy-finite functions. Spaces of harmonic functions are mainly discussed in §6. In §7, we consider a functional completion to define those functions which correspond to continuous BLD-functions in the classical theory (cf. [12], [5]). Finally in §8, we shall show that some part of the theory of Royden boundary (cf. [29], [10]) can be also developed in the axiomatic theory and a Neumann problem can be discussed (cf. [19], [20] for the classical case).

Presentations of these lectures are almost self-contained. The biggest exception is that we use without proof the existence of Green functions and the integral representation theorem for potentials on Brelot's harmonic spaces. For these one may refer to [16] and [11]. Some examples are given without detailed explanations. In the Appendix, networks are studied as examples of harmonic spaces.

Terminology and notation

Given a topological space X and a subset A of X, we denote by $\bar{A}$ the closure of A, $\mathring{A}$ the interior of A and ∂A the boundary of A. For two sets A,B, $A\setminus B$ means the difference set. The family of all open subsets of X is denoted by $\mathcal{O}_X$. A connected open set is called a domain. By a function, we shall always mean an extended real valued function. A continuous function will mean a finite-valued one. The set of all continuous functions on X is denoted by $\mathcal{C}(X)$, and the set of all $f\in \mathcal{C}(X)$ having compact supports in X is denoted by $\mathcal{C}_0(X)$. The support of f is denoted by Supp f. Given a set $A\subset X$ and a class $\mathcal{F}$ of functions on A, we say that $\mathcal{F}$ separates points of A if for any $x,y\in A$, $x\neq y$, there are $f,g\in\mathcal{F}$ satisfying $f(x)g(y) \neq f(y)g(x)$ (with convention $0\cdot\infty = \infty\cdot 0 = 0$). For two classes $\mathcal{F}_1$, $\mathcal{F}_2$ of finite valued functions, $\mathcal{F}_1-\mathcal{F}_2 = \{f_1-f_2 \mid f_1\in\mathcal{F}_1, f_2\in\mathcal{F}_2\}$. For a class $\mathcal{F}$ of functions, $\mathcal{F}^+=\{f\in\mathcal{F} \mid f \geq 0\}$.

For a locally compact space X, a measure on X will mean a (signed) real Radon measure on X. The set of all measures on X is denoted by $\mathcal{M}(X)$. For $\mu\in\mathcal{M}(X)$, μ^+ and μ^- denotes the positive part and the negative part of μ, and $|\mu| = \mu^+ + \mu^-$. For $\mu \in \mathfrak{M}(X)$ and $f \in \mathcal{C}(X)$, $f\mu$ is the measure defined by $(f\mu)(\varphi) = \mu(f\varphi)$ for $\varphi \in \mathcal{C}_0(X)$. Restriction of a function or a measure to a set A is denoted by $\cdot|A$.

By a <u>sheaf</u> of functions on X (resp. a sheaf of measures on X), we mean a mapping Φ defined on $\mathcal{O}_X$ satisfying the following three conditions:

(a) for any $U \in \mathcal{O}_X$, $\Phi(U)$ is a set of functions (resp. measures) on U;

(b) if $U, V \in \mathcal{O}_X$, $U \subset V$ and $\varphi \in \Phi(V)$, then $\varphi|V \in \Phi(U)$;

(c) if $(U_\iota)_{\iota \in I}$ is a subfamily of $\mathcal{O}_X$, φ is a function (resp. measure) on $\bigcup_{\iota \in I} U_\iota$ and if $\varphi|U_\iota \in \Phi(U_\iota)$ for all $\iota \in I$, then $\varphi \in \Phi(\bigcup_{\iota \in I} U_\iota)$.

The mapping $\mathfrak{M}$: $U \mapsto \mathfrak{M}(U)$ is a sheaf, which is called <u>the sheaf of measures</u> on X.

For a locally compact space X with a countable base, a sequence $\{U_n\}$ of relatively compact open sets U_n such that $\overline{U}_n \subset U_{n+1}$ for each n and $\bigcup U_n = X$ is called an <u>exhaustion</u> of X.