Undergraduate Texts in Mathematics

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(continued after index)

Jack Macki Aaron Strauss

Introduction to Optimal Control Theory

With 70 Figures

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Dedicated to the memory of Aaron Strauss
(1940–1977)

Preface

This monograph is an introduction to optimal control theory for systems governed by vector ordinary differential equations. It is not intended as a state-of-the-art handbook for researchers. We have tried to keep two types of reader in mind: (1) mathematicians, graduate students, and advanced undergraduates in mathematics who want a concise introduction to a field which contains nontrivial interesting applications of mathematics (for example, weak convergence, convexity, and the theory of ordinary differential equations); (2) economists, applied scientists, and engineers who want to understand some of the mathematical foundations of optimal control theory.

In general, we have emphasized motivation and explanation, avoiding the "definition-axiom-theorem-proof" approach. We make use of a large number of examples, especially one simple canonical example which we carry through the entire book. In proving theorems, we often just prove the simplest case, then state the more general results which can be proved. Many of the more difficult topics are discussed in the "Notes" sections at the end of chapters and several major proofs are in the Appendices. We feel that a solid understanding of basic facts is best attained by at first avoiding excessive generality.

We have not tried to give an exhaustive list of references, preferring to refer the reader to existing books or papers with extensive bibliographies. References are given by author's name and the year of publication, e.g., Waltman [1974].

Prerequisites for reading this monograph are basic courses in ordinary differential equations, linear algebra, and modern advanced calculus (including some Lebesgue integration). Some functional analysis is used, but the proofs involved may be treated as optional. We have summarized the relevant facts from these areas in an Appendix. We also give references in this Appendix to standard texts in these areas.

We would like to express our appreciation to: Professor Jim Yorke of the University of Maryland for providing several important and original proofs to simplify the presentation of difficult material; Dr. Stephen Lewis of the University of Alberta for providing several interesting examples from Economics; Ms. Peggy Gendron of the University of Minnesota and June Talpash and Laura Thompson of Edmonton, Alberta for their excellent typing work; the universities (and, ultimately, the relevant taxpayers) of Alberta, Maryland, and Minnesota – the first two for their direct financial support, and the last for providing facilities for J.W.M. while on sabbatical; The National Research Council of Canada, for its continuing support of J.W.M.

Edmonton, Alberta
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Jack W. Macki

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Contents

List of Symbols

$\mathcal{B}(\mathbf{x};\alpha)$	the open ball about $\mathbf{x}$ of radius α, $\{\mathbf{y} \mid \|\mathbf{y}-\mathbf{x}\|<\alpha\}$
C, $C[\mathbf{u}(\cdot)]$	cost function
$C(\mathbf{x}_0)$	cost function evaluated along the optimal control and response from $\mathbf{x}_0$
C^1	class of functions having continuous first partial derivatives
C^∞	functions having continuous partials of every order
C_0^∞	functions from C^∞ which have compact support
$\mathrm{co}(\Omega)$	convex hull of Ω
$\mathscr{C}$	controllable set
$\mathscr{C}(t)$	controllable set at time t
$\mathscr{C}_{\mathrm{BB}}$	controllable set using bang–bang controls
$\mathscr{C}_{\mathrm{BB}}(t)$	controllable set at time t using bang–bang controls
$\mathscr{C}_{\mathrm{BBPC}}$	controllable set using bang–bang piecewise constant controls
$d(\mathbf{x}, P)$	$\inf\{\|\mathbf{x}-\mathbf{y}\|: \mathbf{y}\in P\}$
∂S	boundary of the set S
f^0	$C[u(\cdot)]=\int_0^t f^0(s, \mathbf{x}[s], \mathbf{u}(s))\, ds$
$\mathbf{f}_{\mathbf{x}}$	matrix of partials $\partial f^i/\partial x^j$
$\hat{\mathbf{f}}$	$(f^0, \mathbf{f}^T)^T$; the extended velocity vector
$\mathrm{grad}_{\mathbf{x}} H$	gradient $=\left(\dfrac{\partial H}{\partial x^1}, \ldots, \dfrac{\partial H}{\partial x^n}\right)$

$h(P, Q)$	Hausdorff metric $= \inf\{\varepsilon : P \subset N(Q, \varepsilon)$ and $Q \subset N(P, \varepsilon)\}$
$H, H(\hat{\mathbf{w}}, \mathbf{x}, \mathbf{u})$	$\langle \hat{\mathbf{w}}, \hat{\mathbf{f}}(\mathbf{x}, \mathbf{u})\rangle$
Int S	the interior of the set S
$k(\tau)$	$\left\{\sum_{i=1}^{P} c^i[\hat{\mathbf{f}}(x_*[\tau], \mathbf{v}_i) - \hat{\mathbf{f}}(x_*[\tau], u_*(\tau))]c^i \geq 0, v_i \in \Psi, P \in N\right\}$; here $(x_*[\cdot], y_*(\cdot))$ is optimal and Ψ is the range set for admissible controls
$K(t; \mathbf{x}_0)$	reachable set at time t
$K_{BB}(t; \mathbf{x}_0)$	reachable set at time t using bang-bang controls
$\mathcal{K}(t)$	$\left\{\sum_{i=1}^{P} c^i Y(t, \tau_i)\, \hat{\mathbf{z}}_i \mid c_i \geq 0, \hat{z}_i \in k(\tau_i)\right\}$, where $Y(t, \tau_i)$ is the fundamental matrix for (Lin) satisfying $Y(\tau_i, \tau_i) = I$
$\bar{\mathcal{K}}(t_1)$	$\{\hat{\mathbf{a}} + \beta\hat{\mathbf{b}} \mid \lvert\beta\rvert \leq \beta_0, \hat{\mathbf{a}} \in \mathcal{K}(t_1), \hat{\mathbf{b}} = \hat{\mathbf{f}}(\mathbf{x}_*[t_1], \mathbf{u}_*(t_1))\}$
(L)	$\dot{\mathbf{x}} = A(t)\mathbf{x} + B(t)\mathbf{u} + \mathbf{c}(t)$
(LA)	$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$
m	dimension of control vectors $\mathbf{u}(t)$
$M(\hat{\mathbf{w}}, \mathbf{x})$	$\sup\{H(\hat{\mathbf{w}}, \mathbf{x}, \mathbf{v}) : \mathbf{v} \in \Omega\}$
$M \equiv (B, AB, \ldots, A^{n-1}B)$	the controllability matrix for (LA)
$N(P, \varepsilon)$	$\{\mathbf{x} : d(\mathbf{x}, P) < \varepsilon\}$
$o(\lvert\mathbf{x}\rvert)$	stands for any function $h(\mathbf{x})$ such that $\lim_{\mathbf{x}\to 0} \frac{h(\mathbf{x})}{\lvert\mathbf{x}\rvert} = 0$
$Q^+(t, \mathbf{x})$	$\{(y^0, \mathbf{y}) \mid \exists \mathbf{v} \in \Omega, \mathbf{y} = \mathbf{f}(t, \mathbf{x}, \mathbf{v}), y_0 \geq f^0(t, \mathbf{x}, \mathbf{v})\}$
RC	reachable cone, $\bigcup_{t>t_0} (t, K(t, \mathbf{x}_0))$
R^n	Euclidean n-dimensional space
sgn α	$\alpha/\lvert\alpha\rvert$ provided $\alpha \neq 0$
$\mathcal{T}(t)$	target state
$\mathcal{U}_{BB}$	class of functions in $\mathcal{U}_m$ for which $\lvert u^i(t)\rvert = 1$
$\mathcal{U}_m$	$\bigcup_{t_1>t_0} \mathcal{U}_m(t_0, t_1)$
$\mathcal{U}_m(t_0, t_1)$	class of measurable functions from $[t_0, t_1]$ to Ω
$\mathcal{U}_{PC}$	class of piecewise constant functions in $\mathcal{U}_m$
$\mathcal{U}_{PS}$	class of piecewise smooth functions in $\mathcal{U}_m$
$\mathcal{U}_r$	class of piecewise constant functions in $\mathcal{U}_m$ with at most r discontinuities

$\mathcal{U}_\lambda$	class of functions in $\mathcal{U}_m$ having Lipschitz constant λ
$\mathcal{V}_m(t_0, t_1)$	class of measurable functions from $[t_0, t_1]$ to a given bounded set $\Psi \subset R^m$
$\mathcal{V}_m$	$\bigcup_{t_1>t_0} \mathcal{V}_m(t_0, t_1)$
$\mathbf{x}(t; t_0, \mathbf{x}_0, \mathbf{u}(\cdot))$	solution of relevant differential equation through $\mathbf{x}_0$ at time t_0 corresponding to $\mathbf{u}(\cdot)$; the *state* vector
x^i	i^{th} component of $\mathbf{x}$
$\mathbf{x}^T$	transpose of $\mathbf{x}$
$\hat{\mathbf{x}}$	$(x^0, \mathbf{x}) \in R^{n+1}$; the extended state vector
$\langle \mathbf{x}, \mathbf{y} \rangle$	$\sum_i x^i y^i$
Δ	class of successful controls: they steer the initial state to the target
χ_Q	characteristic function of a set Q, i.e., $\chi = +1$ on Q, 0 on the complement of Q
Ω	the unit cube in R^m