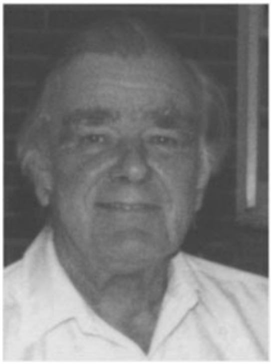
Classics in Mathematics

Roger C. Lyndon • Paul E. Schupp

Combinatorial Group Theory



Roger Lyndon, born on Dec. 18, 1917 in Calais (Maine, USA), entered Harvard University in 1935 with the aim of studying literature and becoming a writer. However, when he discovered that, for him, mathematics required less effort than literature, he switched and graduated from Harvard in 1939. After completing his Master's Degree in 1941, he taught at Georgia Tech, then returned to Harvard in 1942 and there taught navigation to pilots while, supervised by S. MacLane, he studied for his Ph.D., awarded in 1946 for a thesis entitled *The Cohomology Theory of Group Extensions*.

Influenced by Tarski, Lyndon was later to work on model theory. Accepting a position at Princeton, Ralph Fox and Reidemeister's visit in 1948 were major influencea on him to work in combinatorial group theory. In 1953 Lyndon left Princeton for a chair at the University of Michigan where he then remained except for visiting professorships at Berkeley, London, Montpellier and Amiens. Lyndon made numerous major contributions to combinatorial group theory. These included the development of "small cancellation theory", his introduction of "aspherical" presentations of groups and his work on length functions. He died on June 8, 1988.

Paul Schupp, born on March 12, 1937 in Cleveland, Ohio was a student of Roger Lyndon's at the University of Michigan where he wrote a thesis of "Dehn's Algorithm and the Conjugacy Problem". After a year at the University of Wisconsin he moved to the University of Illinois where he remained. For several years he was also concurrently Visiting Professor at the University Paris VII and a member of the Laboratoire d'Informatique Théorique et Programmation (founded by M. P. Schutzenberger).

Schupp further developed the use of cancellation diagrams in combinatorial group theory, introducing conjugacy diagrams, diagrams on compact surfaces, diagrams over free products with amalgamation and HNN extensions and applications to Artin groups. He then worked with David Muller on connections between group theory and formal language theory and on the theory of finite automata on infinite inputs. His current interest is using geometric methods to investigate the computational complexity of algorithms in combinatorial group theory.

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To Wilhelm Magnus

Preface

The first formal development of group theory, centering around the ideas of Galois, was limited almost entirely to finite groups. The idea of an abstract infinite group is clearly embodied in the work of Cayley on the axioms for a group, but was not immediately pursued to any depth. There developed later a school of group theory, in which Schmidt was prominent, that was concerned in part with developing for infinite groups results parallel to those known for finite groups. Another strong influence on the development of group theory was the recognition, notably by Klein, of the role of groups, many of them infinite, in geometry, as well as the development of continuous groups initiated by Lie. A major stimulus to the study of infinite discontinuous groups was the development of topology: we mention particularly the work of Poincaré, Dehn, and Nielsen. This last influence is especially important in the present context since it led naturally to the study of groups presented by generators and relations.

Recent years have seen a steady increase of interest in infinite discontinuous groups, both in the systematic development of the abstract theory and in applications to other areas. The connections with topology have continued to grow. Since Novikov and Boone exhibited groups with unsolvable word problem, results in logic and decision problems have had a great influence on the subject of infinite groups, and through this connection on topology.

Important contributions to the development of the ideas initiated by Dehn were made by Magnus, who has in turn been one of the strongest influences on contemporary research. The book *Combinatorial Group Theory*, by Magnus, Karrass, and Solitar, which appeared in 1966 and immediately became the classic in its field, was dedicated to Dehn. It is our admiration for that work which has prompted us to give this book the same title. We hope that our intention has been realized of taking a further step towards a systematic and comprehensive exposition and survey of the subject.

We view the area of combinatorial group theory as adequately delineated by the book of Magnus, Karrass, and Solitar. It is not necessary for us to list here the topics we discuss, which can be seen from the table of contents. However, we would like to note that there are two broad methods running through our treatment. The first is the 'linear' cancellation method of Nielsen, which plays an important role in Chapters I and IV; this is concerned with the formal expression of an element of a group in terms of a given set of generators for the group. The second is the more geometric method, initiated by Poincaré and Dehn, which includes many of the more recent developments in 'small cancellation theory'; this method, which plays a role in Chapters II, III, and especially V, concerns the formal expression of an element of a normal subgroup N of a group G in terms of conjugates of a given set of elements whose normal closure in G is N.

We have put considerable emphasis on connections with topology, on arguments of a primitive geometric nature, and on connections with logic. In our presentation we have tried to combine a fairly self-contained exposition at a modest level with a reasonably adequate source of reference on the topics discussed. This, together with the fact that the individual chapters were written separately by the two authors, although in close collaboration, has led to considerable variation of style, which we have nonetheless sought to adapt to the subject matter.

While we do not feel it necessary to defend our inclusions, we do feel some need to justify our omissions. There are, of course, many important branches of group theory, for example, most of the theory of finite groups, that no one would claim as part of combinatorial group theory. A borderline area, with which we have made no attempt to deal here, is that of infinite groups subject to some kind of finiteness condition. Beyond these there remain a number of important topics that we believe do belong to combinatorial group theory, but which we have mentioned only briefly if at all, on the grounds that we could not hope to improve on existing excellent treatments of these topics. We list some of these topics.

1. *Commutator calculus and Lie theory.* An excellent treatment is given in Chapter 5 of the book of Magnus, Karrass, Solitar (1966). The 'Alberta notes' of Philip Hall have been republished in 1970.

2. *Varieties of groups.* The definitive work here is the book of H. Neumann (1967).

3. *Linear groups.* Treatments germaine to our topic are given by Dixon (1973) and by Wehrfritz (1973).

4. *Groups acting on trees.* This powerful method of Bass and Serre is central to our topic. An account of this theory is contained in the widely circulated notes of Serre (1968/1969), which are intended to appear in the Springer Lecture Notes series.

5. *Ends of groups.* The development of this subject by Stallings (1968, 1968, 1970, 1971) and Swan (1969) is also central to our subject; a lucid and comprehensive account, from a somewhat different point of view, is given in the book of Cohen (1972).

6. *Cohomology theory.* Of a number of excellent sources, the book of Gruenberg (1970) seems nearest to the spirit of our discussion.

We wish also to draw attention to a few other books that are especially relevant to our topic. For a history of group theory up to the early part of this century we refer to Wussing (1960). The book of Kurosh, in its various editions and translations, remains, along with the book of Magnus, Karrass, and Solitar, the classic source for information on infinite groups. The book of Coxeter and Moser (1965) contains, among other things, presentations for a large number of groups, mainly of geometric origin. We have borrowed much from the book of Zieschang, Vogt, and Coldewey (1970). On the subject of Fuchsian groups from a combinatorial point of view we recommend, in addition to the work just cited, the Dundee notes of Macbeath (1961) and the book of Magnus (1974). For an elementary exposition of the basic connections between topology and group theory we refer to Massey (1967). For a thorough discussion of decision problems in group theory we refer to Miller (1971).

Acknowledgements

The suggestion that such a book as this be written was made in a letter from Springer Editor Peter Hilton, written from Montpellier. It is fitting that the completed manuscript should now be submitted from Montpellier.

The first author (R.C.L.) presented the first draft of some of this work in a seminar at Morehouse College, Atlanta University, in the Fall of 1969. Later versions were developed and presented in lectures at the University of Michigan and during a short visit at Queen Mary College, University of London. Much of the work was done at the Université des Sciences et Techniques du Languedoc, Montpellier, in the year 1972/73 and in the present year, and, for a shorter period, at the University of Birmingham. He is grateful for the hospitality of these universities, and also the Ruhr-Universität Bochum. He gratefully acknowledges support of the National Science Foundation (U.S.A.) and the Science Research Council (U.K.).

The second author (P.E.S.) is grateful to the University of Illinois for an appointment to the Center for Advanced Study, University of Illinois, during the academic year 1973/74. He is also grateful for the hospitality of Queen Mary College, London, and to the University of Manitoba for various periods during the preparation of this book.

We are both greatly indebted to colleagues and students, both at the universities named above and elsewhere, for discussion and criticism. For help with the manuscript we are grateful to Mme. Barrière and to Mrs. Maund. For great help and patience with editorial matters we are grateful to Dr. Alice Peters and to Roberto Minio of Springer-Verlag.

Postscript, February 1977. We have taken advantage of the time before going to press to bring the manuscript more up to date by adding a few new passages in the text and by substantial additions to the bibliography.

R.C.L., Montpellier 1974
P.E.S., Urbana 1974

Table of Contents

Notation

We have tried to use only standard notation, and list below only a few usages that might offer difficulty.

Set theory

$\varnothing$ is the empty set.
$X - Y$ is set difference, where Y is contained in X.
$X + Y$ is union, where X and Y are disjoint.
$\{x_1, \ldots, x_n\}$ is the unordered n-tuple, $(x_1, \ldots, x_n)$ the ordered n-tuple; when there is no ambiguity we write $x_1, \ldots, x_n$ for either.
$X \subset Y$ or $X \subseteq Y$ denotes inclusion, proper or not; $X \subsetneqq Y$ denotes strict inclusion.
$|X|$ denotes the cardinal of the set X (except in special contexts).

General

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ denote the (non negative) natural numbers, the integers, the rationals, the reals, the complex numbers.
$\mathbb{GL}(n, K)$, $\mathbb{SL}(n, K)$, $\mathbb{PL}(n, K)$, $\mathbb{PSL}(n, K)$ denote the general, special, projective, and projective special linear group of degree n over the ring K.

Group theory

1 denotes the trivial group, $\mathbb{Z}$(or C) the infinite cyclic group, $\mathbb{Z}_n$(or C_n) the cyclic group of order n.
$\langle U \rangle$ or $Gp(U)$ denotes the subgroup of G generated by the subset U, and according to context, the free group with basis U.
$\langle X; R\rangle$, $(X; R)$, $\langle x_1, \ldots, x_n; r_1, \ldots, r_n\rangle$, as well as several other variants, denote the presentation with generaters $x \in X$ and relators $r \in R$, or the group so presented.
$H < G$ or $H \leq G$ means that H is a subgroup of G.
$H \lhd G$ means that H is a normal subgroup of G.
$|G|$ is the order of G (finite or infinite), except in special contexts.
$|G: H|$ is the index of H in G.

$|w|$, for w an element of a free group with basis X, is the length of w as a reduced word relative to the basis X.

$[h, k] = h^{-1}k^{-1}hk$ (occasionally, where indicated, $hkh^{-1}k^{-1}$).

$[H, K]$ is the subgroup generated by all $[h, k]$ for $h \in H$, $k \in K$.

$C_G(H)$, $N_G(U)$ are the centralizer and normalizer in G of the subset U.

G_p or $\mathrm{Stab}_G(p)$ is the stabilizer of p under action of G.

Aut G is the automorphism group of G.

$G \times H$ is the direct product.

$G * H$, $*\{G_i : i \in I\}$, or $*G_i$ denotes the free product.

$G \underset{A}{*} H$ denotes the free product of G and H with $A = G \cap H$ amalgamated; $\langle G, H; A = B, \phi\rangle$ denotes the free product of (disjoint) groups G and H with their subgroup A and B amalgamated according to the isomorphism $\phi: A \to B$.

$\langle G, t; t^{-1}at = \phi(a), a \in A\rangle$ denotes the indicated HNN extension of G.

Transformations that occur as elements of groups will ordinarily be written on the right: $x \mapsto xT$; other functions will occasionally be written on the left, e.g., $\chi(G)$ for the characteristic function of a group G.

Note on Format

The notation (I.2.3) refers to Proposition 2.3 of Chapter I (to be found in section 2). Similarly, (I.2) refers to that section, and (I) to Chapter I.

A date accompanying a name, e.g., Smith, 1970, refers to a paper or book listed in the bibliography.

A proof begins and ends with the mark □. This mark immediately following the statement of a proposition means that no (further) proof will be given.