Alexander Lubotzky
Dan Segal

Subgroup Growth

Birkhäuser Verlag
Basel · Boston · Berlin

Authors:

Alexander Lubotzky
Institute of Mathematics
Hebrew University
Jerusalem 91904
Israel
e-mail: alexlub@math.huji.ac.il

Dan Segal
Mathematical Institute
All Souls College
Oxford OX1 4AL
UK
e-mail: dan.segal@all-souls.ox.ac.uk

2000 Mathematics Subject Classification 20E07

A CIP catalogue record for this book is available from the Library of Congress, Washington D.C., USA

Bibliographic information published by Die Deutsche Bibliothek
Die Deutsche Bibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data is available in the Internet at <http://dnb.ddb.de>.

ISBN-13: 978-3-0348-9846-1 e-ISBN-13: 978-3-0348-8965-0
DOI: 10.1007/978-3-0348-8965-0

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, re-use of illustrations, broadcasting, reproduction on microfilms or in other ways, and storage in data banks. For any kind of use whatsoever, permission from the copyright owner must be obtained.

© 2003 Birkhäuser Verlag, P.O. Box 133, CH-4010 Basel, Switzerland

Softcover reprint of the hardcover 1st edition 2003

Member of the BertelsmannSpringer Publishing Group
Printed on acid-free paper produced of chlorine-free pulp. TCF ∞

9 8 7 6 5 4 3 2 1 www.birkhauser.ch

Contents

Preface xv

Notation xxi

0 Introduction and Overview 1
0.1 Preliminary comments and definitions 2
0.2 Overview of the chapters 3
0.3 On CFSG 8
0.4 The windows 9
0.5 The 'notes' 9

1 Basic Techniques of Subgroup Counting 11
1.1 Permutation representations 12
1.2 Quotients and subgroups 13
1.3 Group extensions 14
1.4 Nilpotent and soluble groups 20
1.5 Abelian groups I 22
1.6 Finite p-groups 24
1.7 Sylow's theorem 26
1.8 Restricting to soluble subgroups 28
1.9 Applications of the 'minimal index' 29
1.10 Abelian groups II 31
1.11 Growth types 34
Notes 36

2 Free Groups 37
2.1 The subgroup growth of free groups 39
2.2 Subnormal subgroups 42
2.3 Counting d-generator finite groups 43
Notes 50

3 Groups with Exponential Subgroup Growth 51
3.1 Upper bounds 55
3.2 Lower bounds 58
3.3 Free pro-p groups 61

3.4 Normal subgroups in free pro-p groups . . . 63
3.5 Relations in p-groups and Lie algebras . . . 69
Notes . . . 72

4 Pro-p Groups . . . 73
4.1 Pro-p groups with polynomial subgroup growth . . . 74
4.2 Pro-p groups with slow subgroup growth . . . 77
4.3 The groups $\mathrm{SL}_r^1(\mathbb{F}_p[[t]])$ . . . 80
4.4 Λ-perfect groups . . . 83
4.5 The Nottingham group . . . 86
4.6 Finitely presented pro-p groups . . . 86
Notes . . . 90

5 Finitely Generated Groups with Polynomial Subgroup Growth . . . 91
5.1 Preliminary observations . . . 94
5.2 Linear groups with PSG . . . 96
5.3 Upper chief factors . . . 98
5.4 Groups of prosoluble type . . . 101
5.5 Groups of finite upper rank . . . 102
5.6 The degree of polynomial subgroup growth . . . 104
Notes . . . 108

6 Congruence Subgroups . . . 111
6.1 The characteristic 0 case . . . 115
6.2 The positive characteristic case . . . 121
6.3 Perfect Lie algebras . . . 125
6.4 Normal congruence subgroups . . . 128
Notes . . . 132

7 The Generalized Congruence Subgroup Problem . . . 133
7.1 The congruence subgroup problem . . . 136
7.2 Subgroup growth of lattices . . . 141
7.3 Counting hyperbolic manifolds . . . 149
Notes . . . 151

8 Linear Groups . . . 153
8.1 Subgroup growth, characteristic 0 . . . 155
8.2 Residually nilpotent groups . . . 156
8.3 Subgroup growth, characteristic p . . . 156
8.4 Normal subgroup growth . . . 158
Notes . . . 160

9 Soluble Groups . . . 161
9.1 Metabelian groups . . . 162
9.2 Residually nilpotent groups . . . 164
9.3 Some finitely presented metabelian groups . . . 167
9.4 Normal subgroup growth in metabelian groups . . . 172
Notes . . . 175

10 Profinite Groups with Polynomial Subgroup Growth 177
10.1 Upper rank 180
10.2 Profinite groups with wPSG: structure 181
10.3 Quasi-semisimple groups 187
10.4 Profinite groups with wPSG: characterization 194
10.5 Weak PSG = PSG 197
Notes 200

11 Probabilistic Methods 201
11.1 The probability measure 205
11.2 Generation probabilities 207
11.3 Maximal subgroups 209
11.4 Further applications 211
11.5 Pro-p groups 214
Notes 217

12 Other Growth Conditions 219
12.1 Rank and bounded generation 225
12.2 Adelic groups 226
12.3 The structure of finite linear groups 229
12.4 Composition factors 230
12.5 BG, PIG and subgroup growth 233
12.6 Residually nilpotent groups 234
12.7 Arithmetic groups and the CSP 236
12.8 Examples 237
Notes 241

13 The Growth Spectrum 243
13.1 Products of alternating groups 244
13.2 Some finitely generated permutation groups 248
13.3 Some profinite groups with restricted composition factors . . 255
13.4 Automorphisms of rooted trees 260
Notes 266

14 Explicit Formulas and Asymptotics 269
14.1 Free groups and the modular group 269
14.2 Free products of finite groups 271
14.3 Modular subgroup arithmetic 274
14.4 Surface groups 277
Notes 283

15 Zeta Functions I: Nilpotent Groups 285
15.1 Local zeta functions as p-adic integrals 290
15.2 Alternative methods 297
15.3 The zeta function of a nilpotent group 304
Notes 307

16 Zeta Functions II: p-adic Analytic Groups 309
16.1 Integration on pro-p groups 312
16.2 Counting subgroups in a p-adic analytic group 313
16.3 Counting orbits 315
16.4 Counting p-groups 316
Notes 318

Windows

1 Finite Group Theory 319
1 Hall subgroups and Sylow bases 319
2 Carter subgroups 320
3 The Fitting subgroup 320
4 The generalized Fitting subgroup 322
5 Tate's theorem 322
6 Rank and p-rank 323
7 Schur multiplier 323
8 Powerful p-groups 324
9 GL_n and $\mathrm{Sym}(n)$ 325

2 Finite Simple Groups 329
1 The list 330
2 Generators 331
3 Subgroups 332
4 Representations 332
5 Automorphisms 334
6 Schur multipliers 335
7 An elementary proof 335

3 Permutation Groups 337
1 Primitive groups 337
2 Groups with restricted sections 338
3 Subgroups of alternating groups 343

4 Profinite Groups 349
1 Completions 350
2 Free profinite groups 352
3 Profinite presentations 353

5 Pro-p Groups 357
1 Generators and relations 357
2 Pro-p groups of finite rank 359
3 Linear pro-p groups over local fields 361
4 Automorphisms of finite p-groups 363
5 Hall's enumeration principle 364

6 Soluble Groups . . . 367
1 Nilpotent groups . . . 367
2 Soluble groups of finite rank . . . 369
3 Finitely generated metabelian groups . . . 372

7 Linear Groups . . . 375
1 Soluble groups . . . 375
2 Jordan's theorem . . . 376
3 Monomial groups . . . 376
4 Finitely generated groups . . . 376
5 Lang's theorem . . . 377

8 Linearity Conditions for Infinite Groups . . . 379
1 Variations on Mal'cev's local theorem . . . 379
2 Groups that are residually of bounded rank . . . 383
3 Applications of Ado's theorem . . . 385

9 Strong Approximation for Linear Groups . . . 389
1 A variant of the Strong Approximation Theorem . . . 390
2 Subgroups of $\mathrm{SL}_n(\mathbb{F}_p)$ . . . 397
3 The 'Lubotzky alternative' . . . 400
4 Strong approximation in positive characteristic . . . 406

10 Primes . . . 409
1 The Prime Number Theorem . . . 409
2 Arithmetic progressions and the Bombieri-Vinogradov theorem . . . 411
3 Global fields and Chebotarev's theorem . . . 413

11 Probability . . . 415

12 p-adic Integrals and Logic . . . 419
1 Results . . . 419
2 A peek inside the black box . . . 421

Open Problems . . . 425
1 'Growth spectrum' . . . 425
2 Normal subgroup growth in pro-p groups and metabelian groups . . . 428
3 The degree of f.g. nilpotent groups . . . 428
4 Finite extensions . . . 429
5 Soluble groups . . . 429
6 Isospectral groups . . . 429
7 Congruence subgroups, lattices in Lie groups . . . 430
8 Other growth conditions . . . 430
9 Zeta functions . . . 431

Bibliography . . . 433

Index . . . 446