

# **Optimization in Elliptic Problems with Applications to Mechanics of Deformable Bodies and Fluid Mechanics**

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