

ANALYSIS

SECOND EDITION

Elliott H. Lieb

Princeton University

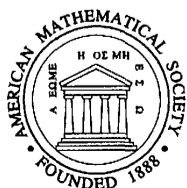
Michael Loss

Georgia Institute of Technology

Graduate Studies

in Mathematics

Volume 14



American Mathematical Society
Providence, Rhode Island

Editorial Board

Lance Small (Chair)
James E. Humphreys
Julius L. Shaneson
David Sattinger

2000 *Mathematics Subject Classification*. Primary 28-01, 42-01, 46-01, 49-01;
Secondary 26D10, 26D15, 31B05, 31B15, 46E35, 46F05, 46F10, 49-XX, 81Q05.

ABSTRACT. This book is a course in real analysis that begins with the usual measure theory yet brings the reader quickly to a level where a wider than usual range of topics can be appreciated, including some recent research. The reader is presumed to know only basic facts learned in a good course in calculus. Topics covered include L^p -spaces, rearrangement inequalities, sharp integral inequalities, distribution theory, Fourier analysis, potential theory and Sobolev spaces. To illustrate the topics, the book contains a chapter on the calculus of variations, with examples from mathematical physics, and concludes with a chapter on eigenvalue problems.

The book will be of interest to beginning graduate students of mathematics, as well as to students of the natural sciences and engineering who want to learn some of the important tools of real analysis.

Library of Congress Cataloging-in-Publication Data

Lieb, Elliott H.

Analysis / Elliott H. Lieb, Michael Loss.—2nd ed.

p. cm. — (Graduate studies in mathematics; v. 14)

Includes bibliographic references and index.

ISBN 0-8218-2783-9 (alk. paper)

1. Mathematical analysis. I. Loss, Michael, 1954- .

II. Title. III. Series.

QA300.L54 2001

515—dc21

2001018215

CIP

Copying and reprinting. Individual readers of this publication, and nonprofit libraries acting for them, are permitted to make fair use of the material, such as to copy a chapter for use in teaching or research. Permission is granted to quote brief passages from this publication in reviews, provided the customary acknowledgment of the source is given.

Republication, systematic copying, or multiple reproduction of any material in this publication is permitted only under license from the American Mathematical Society. Requests for such permission should be addressed to the Assistant to the Publisher, American Mathematical Society, P.O. Box 6248, Providence, Rhode Island 02940-6248. Requests can also be made by e-mail to reprint-permission@ams.org.

First Edition © 1997 by the authors.

Reprinted with corrections 1997.

Second Edition © 2001 by the authors.

Printed in the United States of America.

∞ The paper used in this book is acid-free and falls within the guidelines established to ensure permanence and durability.

10 9 8 7 6 5 4 3 2 1 06 05 04 03 02 01

Contents

Preface to the First Edition	xvii
Preface to the Second Edition	xxi
CHAPTER 1. Measure and Integration	1
1.1 Introduction	1
1.2 Basic notions of measure theory	4
1.3 Monotone class theorem	9
1.4 Uniqueness of measures	11
1.5 Definition of measurable functions and integrals	12
1.6 Monotone convergence	17
1.7 Fatou's lemma	18
1.8 Dominated convergence	19
1.9 Missing term in Fatou's lemma	21
1.10 Product measure	23
1.11 Commutativity and associativity of product measures	24
1.12 Fubini's theorem	25
1.13 Layer cake representation	26
1.14 Bathtub principle	28
1.15 Constructing a measure from an outer measure	29
1.16 Uniform convergence except on small sets	31
1.17 Simple functions and really simple functions	32
1.18 Approximation by really simple functions	34

1.19	Approximation by C^∞ functions	36
	Exercises	37
	CHAPTER 2. L^p-Spaces	41
2.1	Definition of L^p -spaces	41
2.2	Jensen's inequality	44
2.3	Hölder's inequality	45
2.4	Minkowski's inequality	47
2.5	Hanner's inequality	49
2.6	Differentiability of norms	51
2.7	Completeness of L^p -spaces	52
2.8	Projection on convex sets	53
2.9	Continuous linear functionals and weak convergence	54
2.10	Linear functionals separate	56
2.11	Lower semicontinuity of norms	57
2.12	Uniform boundedness principle	58
2.13	Strongly convergent convex combinations	60
2.14	The dual of $L^p(\Omega)$	61
2.15	Convolution	64
2.16	Approximation by C^∞ -functions	64
2.17	Separability of $L^p(\mathbb{R}^n)$	67
2.18	Bounded sequences have weak limits	68
2.19	Approximation by C_c^∞ -functions	69
2.20	Convolutions of functions in dual $L^p(\mathbb{R}^n)$ -spaces are continuous	70
2.21	Hilbert-spaces	71
	Exercises	75
	CHAPTER 3. Rearrangement Inequalities	79
3.1	Introduction	79
3.2	Definition of functions vanishing at infinity	80
3.3	Rearrangements of sets and functions	80
3.4	The simplest rearrangement inequality	82
3.5	Nonexpansivity of rearrangement	83

3.6 Riesz's rearrangement inequality in one-dimension	84
3.7 Riesz's rearrangement inequality	87
3.8 General rearrangement inequality	93
3.9 Strict rearrangement inequality	93
Exercises	95
CHAPTER 4. Integral Inequalities	97
4.1 Introduction	97
4.2 Young's inequality	98
4.3 Hardy–Littlewood–Sobolev inequality	106
4.4 Conformal transformations and stereographic projection	110
4.5 Conformal invariance of the Hardy–Littlewood–Sobolev inequality	114
4.6 Competing symmetries	117
4.7 Proof of Theorem 4.3: Sharp version of the Hardy–Littlewood–Sobolev inequality	119
4.8 Action of the conformal group on optimizers	120
Exercises	121
CHAPTER 5. The Fourier Transform	123
5.1 Definition of the L^1 Fourier transform	123
5.2 Fourier transform of a Gaussian	125
5.3 Plancherel's theorem	126
5.4 Definition of the L^2 Fourier transform	127
5.5 Inversion formula	128
5.6 The Fourier transform in $L^p(\mathbb{R}^n)$	128
5.7 The sharp Hausdorff–Young inequality	129
5.8 Convolutions	130
5.9 Fourier transform of $ x ^{\alpha-n}$	130
5.10 Extension of 5.9 to $L^p(\mathbb{R}^n)$	131
Exercises	133
CHAPTER 6. Distributions	135
6.1 Introduction	135

6.2	Test functions (The space $\mathcal{D}(\Omega)$)	136
6.3	Definition of distributions and their convergence	136
6.4	Locally summable functions, $L^p_{\text{loc}}(\Omega)$	137
6.5	Functions are uniquely determined by distributions	138
6.6	Derivatives of distributions	139
6.7	Definition of $W^{1,p}_{\text{loc}}(\Omega)$ and $W^{1,p}(\Omega)$	140
6.8	Interchanging convolutions with distributions	142
6.9	Fundamental theorem of calculus for distributions	143
6.10	Equivalence of classical and distributional derivatives	144
6.11	Distributions with zero derivatives are constants	146
6.12	Multiplication and convolution of distributions by C^∞ -functions	146
6.13	Approximation of distributions by C^∞ -functions	147
6.14	Linear dependence of distributions	148
6.15	$C^\infty(\Omega)$ is 'dense' in $W^{1,p}_{\text{loc}}(\Omega)$	149
6.16	Chain rule	150
6.17	Derivative of the absolute value	152
6.18	Min and Max of $W^{1,p}$ -functions are in $W^{1,p}$	153
6.19	Gradients vanish on the inverse of small sets	154
6.20	Distributional Laplacian of Green's functions	156
6.21	Solution of Poisson's equation	157
6.22	Positive distributions are measures	159
6.23	Yukawa potential	163
6.24	The dual of $W^{1,p}(\mathbb{R}^n)$	166
	Exercises	167
	CHAPTER 7. The Sobolev Spaces H^1 and $H^{1/2}$	171
7.1	Introduction	171
7.2	Definition of $H^1(\Omega)$	171
7.3	Completeness of $H^1(\Omega)$	172
7.4	Multiplication by functions in $C^\infty(\Omega)$	173
7.5	Remark about $H^1(\Omega)$ and $W^{1,2}(\Omega)$	174
7.6	Density of $C^\infty(\Omega)$ in $H^1(\Omega)$	174
7.7	Partial integration for functions in $H^1(\mathbb{R}^n)$	175
7.8	Convexity inequality for gradients	177

7.9	Fourier characterization of $H^1(\mathbb{R}^n)$	179
•	Heat kernel	180
7.10	$-\Delta$ is the infinitesimal generator of the heat kernel	181
7.11	Definition of $H^{1/2}(\mathbb{R}^n)$	181
7.12	Integral formulas for $(f, p f)$ and $(f, \sqrt{p^2 + m^2} f)$	184
7.13	Convexity inequality for the relativistic kinetic energy	185
7.14	Density of $C_c^\infty(\mathbb{R}^n)$ in $H^{1/2}(\mathbb{R}^n)$	186
7.15	Action of $\sqrt{-\Delta}$ and $\sqrt{-\Delta + m^2} - m$ on distributions	186
7.16	Multiplication of $H^{1/2}$ functions by C^∞ -functions	187
7.17	Symmetric decreasing rearrangement decreases kinetic energy	188
7.18	Weak limits	190
7.19	Magnetic fields: The H_A^1 -spaces	191
7.20	Definition of $H_A^1(\mathbb{R}^n)$	192
7.21	Diamagnetic inequality	193
7.22	$C_c^\infty(\mathbb{R}^n)$ is dense in $H_A^1(\mathbb{R}^n)$	194
	Exercises	195
	CHAPTER 8. Sobolev Inequalities	199
8.1	Introduction	199
8.2	Definition of $D^1(\mathbb{R}^n)$ and $D^{1/2}(\mathbb{R}^n)$	201
8.3	Sobolev's inequality for gradients	202
8.4	Sobolev's inequality for $ p $	204
8.5	Sobolev inequalities in 1 and 2 dimensions	205
8.6	Weak convergence implies strong convergence on small sets	208
8.7	Weak convergence implies a.e. convergence	212
8.8	Sobolev inequalities for $W^{m,p}(\Omega)$	213
8.9	Rellich–Kondrashov theorem	214
8.10	Nonzero weak convergence after translations	215
8.11	Poincaré's inequalities for $W^{m,p}(\Omega)$	218
8.12	Poincaré–Sobolev inequality for $W^{m,p}(\Omega)$	219
8.13	Nash's inequality	220
8.14	The logarithmic Sobolev inequality	223
8.15	A glance at contraction semigroups	225

8.16	Equivalence of Nash's inequality and smoothing estimates	227
8.17	Application to the heat equation	229
8.18	Derivation of the heat kernel via logarithmic Sobolev inequalities	232
	Exercises	235
	CHAPTER 9. Potential Theory and Coulomb Energies	237
9.1	Introduction	237
9.2	Definition of harmonic, subharmonic, and superharmonic functions	238
9.3	Properties of harmonic, subharmonic, and superharmonic functions	239
9.4	The strong maximum principle	243
9.5	Harnack's inequality	245
9.6	Subharmonic functions are potentials	246
9.7	Spherical charge distributions are 'equivalent' to point charges	248
9.8	Positivity properties of the Coulomb energy	250
9.9	Mean value inequality for $\Delta - \mu^2$	251
9.10	Lower bounds on Schrödinger 'wave' functions	254
9.11	Unique solution of Yukawa's equation	255
	Exercises	256
	CHAPTER 10. Regularity of Solutions of Poisson's Equation	257
10.1	Introduction	257
10.2	Continuity and first differentiability of solutions of Poisson's equation	260
10.3	Higher differentiability of solutions of Poisson's equation	262
	CHAPTER 11. Introduction to the Calculus of Variations	267
11.1	Introduction	267
11.2	Schrödinger's equation	269
11.3	Domination of the potential energy by the kinetic energy	270
11.4	Weak continuity of the potential energy	274
11.5	Existence of a minimizer for E_0	275

11.6 Higher eigenvalues and eigenfunctions	278
11.7 Regularity of solutions	279
11.8 Uniqueness of minimizers	280
11.9 Uniqueness of positive solutions	281
11.10 The hydrogen atom	282
11.11 The Thomas–Fermi problem	284
11.12 Existence of an unconstrained Thomas–Fermi minimizer	285
11.13 Thomas–Fermi equation	286
11.14 The Thomas–Fermi minimizer	287
11.15 The capacitor problem	289
11.16 Solution of the capacitor problem	293
11.17 Balls have smallest capacity	296
Exercises	297
CHAPTER 12. More about Eigenvalues	299
12.1 Min-max principles	300
12.2 Generalized min-max	302
12.3 Bound for eigenvalue sums in a domain	304
12.4 Bound for Schrödinger eigenvalue sums	306
12.5 Kinetic energy with antisymmetry	311
12.6 The semiclassical approximation	314
12.7 Definition of coherent states	316
12.8 Resolution of the identity	317
12.9 Representation of the nonrelativistic kinetic energy	319
12.10 Bounds for the relativistic kinetic energy	319
12.11 Large N eigenvalue sums in a domain	320
12.12 Large N asymptotics of Schrödinger eigenvalue sums	323
Exercises	327
List of Symbols	331
References	335
Index	341