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Entire Functions of Several Complex Variables

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Introduction

I – Entire functions of several complex variables constitute an important and original chapter in complex analysis. The study is often motivated by certain applications to specific problems in other areas of mathematics: partial differential equations via the Fourier-Laplace transformation and convolution operators, analytic number theory and problems of transcendence, or approximation theory, just to name a few.

What is important for these applications is to find solutions which satisfy certain growth conditions. The specific problem defines inherently a growth scale, and one seeks a solution of the problem which satisfies certain growth conditions on this scale, and sometimes solutions of minimal asymptotic growth or optimal solutions in some sense.

For one complex variable the study of solutions with growth conditions forms the core of the classical theory of entire functions and, historically, the relationship between the number of zeros of an entire function $f(z)$ of one complex variable and the growth of $|f|$ (or equivalently $\log|f|$) was the first example of a systematic study of growth conditions in a general setting.

Problems with growth conditions on the solutions demand much more precise information than existence theorems. The correspondence between two scales of growth can be interpreted often as a correspondence between families of bounded sets in certain Fréchet spaces. However, for applications it is of utmost importance to develop precise and explicit representations of the solutions.

If we pass from $\mathbb{C}$ to $\mathbb{C}^n$, new problems such as problems of value distribution for holomorphic mappings from $\mathbb{C}^n$ to C^m arise. On the other hand, new techniques are often needed for classical problems to obtain solutions and representations of the solutions. Zeros of entire functions f are no longer isolated points; a measure of the zero set is obtained by the representation of the divisor X_f of f (and more generally of analytic subvarieties) by closed and positive currents, a class of generalized differential forms.

Paradoxally, it is the non-holomorphic objects, the "soft" objects (objets souples in French, see [C]) of complex analysis, principally plurisubharmonic functions and positive closed currents, which are adapted to problems with growth conditions, giving global representations in $\mathbb{C}^n$. Very often properties of the classical (i.e. holomorphic) objects will be derived from properties obtained for the soft objects. Plurisubharmonic functions were introduced in 1942 by K. Oka and P. Lelong. They occur in a natural way from the beginning of this book. Indicators of growth for a class of entire functions f are obtained as upper bounds for $\log|f|$; for $\log|f|$. To solve Cousin's Second Problem, i.e. to find (with growth conditions) an entire function f with given zeros X in $\mathbb{C}^n$, we solve first the general equation $i\partial\bar{\partial}V=\theta$ for a closed and positive current θ; if $\theta=[X]$, the current of integration on X, we then obtain f by $V=\log|f|$. Properties of plurisubharmonic functions appear again in a remarkable (and unexpected) result of H. Skoda (1972): there exists a representation for the analytic subvarieties Y in $\mathbb{C}^n$ of dimension $p(0\leqq p\leqq n-1)$ as the zero set $Y=F^{-1}(0)$ of an entire mapping $F=(f_1,\ldots,f_{n+1})$ such that $\|F\|$ is controlled by the growth of the area of Y. Plurisubharmonic functions obtained from potentials seem well adapted to the construction of global representations in $\mathbb{C}^n$; the method avoids the delicate study of ideals of holomorphic functions vanishing on Y and satisfying growth conditions.

The same methods using the soft object's properties of the current $(i\partial\bar{\partial}V)^p$ and the Monge-Ampère equation for plurisubharmonic functions V are employed for recent results obtained in value distribution theory of holomorphic mappings $\mathbb{C}^n\to\mathbb{C}^m$ or $X\to Y$, two analytic subvarieties in $\mathbb{C}^n$.

II - Before summarizing the content of this book, we would like to make some remarks.

a) We have not sought to give an exhaustive treatment of the subject (problems for $n>1$ are too numerous for a single book). We have tried to introduce the reader to the central problems of current research in this area, essentially that which had led to general methods or new technics. Applications appear only in Chapter 6 (to analytic number theory) and in Chapters 8 and 9 (to functional analysis).

b) On the other hand, we have tried to make the book self-contained. Some knowledge in the theory for one complex variable is required of the reader, as well as on integration, the calculus of differential forms and the theory of distributions. A list of books where the reader can find general results not developped here is given before the bibliography (such references are given by a capital roman letter).

The proofs of complementary results appear in three appendices: Appendix I for general properties of plurisubharmonic functions, Appendix II for the technic of proximate orders Appendix III for the $\bar{\partial}$ resolution for (0, 1) forms with L^2-estimates by Hörmander's method.

c) The importance of analytic representations, particulary for some applications, has made it necessary to give certain calculations *in extenso*. The authors are aware of the technical aspect of some developments given in the book. We recommand that the reader first read over the proof in order to assimilate the general idea before immersing himself in the details of the calculations.

d) The literature on the subject of entire functions is enormous. The bibliography, without pretending to be exhaustive, gives an overview of those areas of current interest. Each chapter has a short historical note which is an attempt to explain the origin of the given results.

III – Chapter 1 gives the basic definitions of the growth scales in $\mathbb{C}^n$, the notion of order and type, the indicator of growth and proximate orders. These classical notions extend trivially to plurisubharmonic functions and to entire functions in $\mathbb{C}^n$. In Chapter 2, we introduce the reader to the fundamental properties of positive differential forms and of positive and closed currents. Chapter 3 studies the solution with growth conditions of the equation $i\partial\bar{\partial}V=\theta$ for θ a positive closed current of type (1, 1) in $\mathbb{C}^n$, from which we deduce for $V=\log|f|$ the solution with growth conditions in $\mathbb{C}^n$ of Cousin's Second Problem and the representation of entire functions with a given zero set. The result for an entire function of finite order in $\mathbb{C}^n$ gives an extension of classical results of J. Hadamard and E. Lindelöf for $n=1$. Chapter 4 studies the class of entire functions f of regular growth. Certain results are given here for the first time. The importance of this study, which is based on the preceeding chapters, is in the numerous applications (Fourier transforms, differential systems) and the possibility of associating the regular growth of $\log|f|$ with the regular distribution of the zero set of f.

Chapter 5 studies the problems of entire maps $F: \mathbb{C}^n \to \mathbb{C}^m$. The first portion is devoted to the development of a representation of an analytic subvariety Y of $\mathbb{C}^n$ as the zero set of an entire map $F: \mathbb{C}^n \to \mathbb{C}^{n+1}$, that is $Y=F^{-1}(0)$, $F=(f_1, \ldots, f_{n+1})$, with control of the growth of the function $\|F\|$. The second part studies the growth of the fibers $F^{-1}(a) \cap B(0, r)$, where $B(0, r)=\{z: \|z\|<r\}$, when $r \to +\infty$. The third part studies the relationship between the growth of the area of an analytic set in $\mathbb{C}^n$ and its trace on linear subspaces of $\mathbb{C}^n$. The cases of slow growth and algebraic growth are also studied.

Chapter 6 gives an example of an application of the methods of the preceeding chapters to a problem in number theory. We show that the set of points of $\mathbb{C}^n$ where certain families of meromorphic functions of finite order take on algebraic values is contained in an algebraic subvariety of $\mathbb{C}^n$ whose degree can be bounded: this famous result of E. Bombieri (1970) gave a very deep and unexpected application of the theory of closed positive currents t and of the number $\nu_t(x)$ (a kind of multiplicity for x on the support of t) to number theory, via a classical method of Siegel and L^2-estimates for the $\bar{\partial}$ operator. The same idea was also fundamental some time later in Siu's Theorem about the structure of closed positive currents.

Chapter 7 establishes the theory of the indicator of growth theorem for entire functions of finite order in $\mathbb{C}^n$: every plurisubharmonic function positively homogeneous of order ρ is the (regularized) indicator of growth function of an entire function of order ρ.

Chapters 8 and 9 concern applications of entire functions to classes of linear operators. Indeed the space $\hat{\mathscr{D}}(\Omega)$ of the Fourier transforms of the distributions defined in a bounded domain Ω of $\mathbb{C}^n$ is a subspace of the space $\mathscr{H}(\mathbb{C}^n)$ of the entire functions in $\mathbb{C}^n$, and many problems characterize classes of distributions in Ω by growth properties of the image in $\mathscr{H}(\mathbb{C}^n)$. This method leads to analytic functionals. The analytic functionals are the elements of the dual space of the space $\mathscr{H}(\Omega)$ of holomorphic functions in Ω, equipped with the topology of uniform convergence on compact subsets of Ω. Chapter 8 gives a study of the Fourier-Borel transform and of the Laplace transform in order to obtain properties for analytic functionals and their supports.

Chapter 9 gives a general treatment of convolution operators in linear spaces of entire functions. New results in particular for the functions of order $\rho<1$ are given as consequences of the techniques developped in preceeding sections of the book.

We use the following system of notations for references: a statement (theorem, lemma, proposition, definition etc.) is given two numbers, the first indicating the chapter in which it is found and the second indicating its position in that chapter. Thus, Theorem 8.23 refers to the 23-rd statement in Chapter 8. Figures within parentheses refer to equations in the text, for instance (4, 18) refers to the eighteenth equation in Chapter 4. Roman numerals I, II, III, refer to the three appendices which are at the end of the book.

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