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continued after Index

Olli Lehto

Univalent Functions and Teichmüller Spaces

With 16 Illustrations



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Preface

This monograph grew out of the notes relating to the lecture courses that I gave at the University of Helsinki from 1977 to 1979, at the Eidgenössische Technische Hochschule Zürich in 1980, and at the University of Minnesota in 1982. The book presumably would never have been written without Fred Gehring's continuous encouragement. Thanks to the arrangements made by Edgar Reich and David Storvick, I was able to spend the fall term of 1982 in Minneapolis and do a good part of the writing there. Back in Finland, other commitments delayed the completion of the text.

At the final stages of preparing the manuscript, I was assisted first by Mika Seppälä and then by Jouni Luukkainen, who both had a grant from the Academy of Finland. I am greatly indebted to them for the improvements they made in the text.

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Helsinki, Finland
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Olli Lehto

Contents

Preface	v
Introduction.....	1
CHAPTER I	
Quasiconformal Mappings.....	4
Introduction to Chapter I.....	4
1. Conformal Invariants.....	5
1.1 Hyperbolic metric	5
1.2 Module of a quadrilateral	7
1.3 Length-area method.....	8
1.4 Rengel's inequality.....	9
1.5 Module of a ring domain	10
1.6 Module of a path family.....	11
2. Geometric Definition of Quasiconformal Mappings.....	12
2.1 Definitions of quasiconformality	12
2.2 Normal families of quasiconformal mappings.....	13
2.3 Compactness of quasiconformal mappings.....	14
2.4 A distortion function	15
2.5 Circular distortion.....	17
3. Analytic Definition of Quasiconformal Mappings.....	18
3.1 Dilatation quotient	18
3.2 Quasiconformal diffeomorphisms.....	19
3.3 Absolute continuity and differentiability.....	20
3.4 Generalized derivatives	21
3.5 Analytic characterization of quasiconformality.....	22
4. Beltrami Differential Equation	23
4.1 Complex dilatation	23

4.2	Quasiconformal mappings and the Beltrami equation	24
4.3	Singular integrals	25
4.4	Representation of quasiconformal mappings	26
4.5	Existence theorem	27
4.6	Convergence of complex dilatations	28
4.7	Decomposition of quasiconformal mappings	29
5.	The Boundary Value Problem	30
5.1	Boundary function of a quasiconformal mapping	30
5.2	Quasisymmetric functions	31
5.3	Solution of the boundary value problem	33
5.4	Composition of Beurling–Ahlfors extensions	34
5.5	Quasi-isometry	35
5.6	Smoothness of solutions	36
5.7	Extremal solutions	37
6.	Quasidisks	38
6.1	Quasicircles	38
6.2	Quasiconformal reflections	39
6.3	Uniform domains	41
6.4	Linear local connectivity	44
6.5	Arc condition	45
6.6	Conjugate quadrilaterals	46
6.7	Characterizations of quasidisks	47

CHAPTER II

	Univalent Functions	50
	Introduction to Chapter II	50
1.	Schwarzian Derivative	51
1.1	Definition and transformation rules	51
1.2	Existence and uniqueness	53
1.3	Norm of the Schwarzian derivative	54
1.4	Convergence of Schwarzian derivatives	55
1.5	Area theorem	58
1.6	Conformal mappings of a disc	59
2.	Distance between Simply Connected Domains	60
2.1	Distance from a disc	60
2.2	Distance function and coefficient problems	61
2.3	Boundary rotation	62
2.4	Domains of bounded boundary rotation	63
2.5	Upper estimate for the Schwarzian derivative	65
2.6	Outer radius of univalence	66
2.7	Distance between arbitrary domains	67
3.	Conformal Mappings with Quasiconformal Extensions	68
3.1	Deviation from Möbius transformations	68
3.2	Dependence of a mapping on its complex dilatation	69
3.3	Schwarzian derivatives and complex dilatations	72
3.4	Asymptotic estimates	73
3.5	Majorant principle	76
3.6	Coefficient estimates	77

4. Univalence and Quasiconformal Extensibility of Meromorphic Functions.	79
4.1 Quasiconformal reflections under Möbius transformations	79
4.2 Quasiconformal extension of conformal mappings	81
4.3 Exhaustion by quasidisks	83
4.4 Definition of Schwarzian domains	83
4.5 Domains not linearly locally connected	84
4.6 Schwarzian domains and quasidisks	86
5. Functions Univalent in a Disc	87
5.1 Quasiconformal extension to the complement of a disc	87
5.2 Real analytic solutions of the boundary value problem	89
5.3 Criterion for univalence	89
5.4 Parallel strips.	90
5.5 Continuous extension	91
5.6 Image of disks	92
5.7 Homeomorphic extension	94
CHAPTER III	
Universal Teichmüller Space	96
Introduction to Chapter III	96
1. Models of the Universal Teichmüller Space	97
1.1 Equivalent quasiconformal mappings.	97
1.2 Group structures	98
1.3 Normalized conformal mappings	99
1.4 Sewing problem	100
1.5 Normalized quasidisks	101
2. Metric of the Universal Teichmüller Space	103
2.1 Definition of the Teichmüller distance	103
2.2 Teichmüller distance and complex dilatation	104
2.3 Geodesics for the Teichmüller metric	105
2.4 Completeness of the universal Teichmüller space	106
3. Space of Quasisymmetric Functions.	108
3.1 Distance between quasisymmetric functions	108
3.2 Existence of a section.	109
3.3 Contractibility of the universal Teichmüller space	109
3.4 Incompatibility of the group structure with the metric	110
4. Space of Schwarzian Derivatives	111
4.1 Mapping into the space of Schwarzian derivatives	111
4.2 Comparison of distances	112
4.3 Imbedding of the universal Teichmüller space	113
4.4 Schwarzian derivatives of univalent functions.	115
4.5 Univalent functions and the universal Teichmüller space	116
4.6 Closure of the universal Teichmüller space.	116
5. Inner Radius of Univalence	118
5.1 Definition of the inner radius of univalence	118
5.2 Isomorphic Teichmüller spaces	119
5.3 Inner radius and quasiconformal extensions.	120

5.4 Inner radius and quasiconformal reflections	121
5.5 Inner radius of sectors	122
5.6 Inner radius of ellipses and polygons	125
5.7 General estimates for the inner radius	126
CHAPTER IV	
Riemann Surfaces	128
Introduction to Chapter IV	128
1. Manifolds and Their Structures	129
1.1 Real manifolds.	129
1.2 Complex analytic manifolds.	130
1.3 Border of a surface.	131
1.4 Differentials on Riemann surfaces	132
1.5 Isothermal coordinates	133
1.6 Riemann surfaces and quasiconformal mappings	134
2. Topology of Covering Surfaces.	135
2.1 Lifting of paths	135
2.2 Covering surfaces and the fundamental group	136
2.3 Branched covering surfaces	137
2.4 Covering groups	138
2.5 Properly discontinuous groups	140
3. Uniformization of Riemann Surfaces	142
3.1 Lifted and projected conformal structures	142
3.2 Riemann mapping theorem	143
3.3 Representation of Riemann surfaces.	144
3.4 Lifting of continuous mappings	145
3.5 Homotopic mappings	146
3.6 Lifting of differentials.	147
4. Groups of Möbius Transformations.	149
4.1 Covering groups acting on the plane	149
4.2 Fuchsian groups	150
4.3 Elementary groups.	151
4.4 Kleinian groups	152
4.5 Structure of the limit set.	153
4.6 Invariant domains	155
5. Compact Riemann Surfaces	157
5.1 Covering groups over compact surfaces	157
5.2 Genus of a compact surface	158
5.3 Function theory on compact Riemann surfaces	158
5.4 Divisors on compact surfaces.	159
5.5 Riemann–Roch theorem	160
6. Trajectories of Quadratic Differentials	161
6.1 Natural parameters	161
6.2 Straight lines and trajectories.	163
6.3 Orientation of trajectories	164
6.4 Trajectories in the large	165

6.5	Periodic trajectories	166
6.6	Non-periodic trajectories	166
7.	Geodesics of Quadratic Differentials.	168
7.1	Definition of the induced metric	168
7.2	Locally shortest curves.	169
7.3	Geodesic polygons.	170
7.4	Minimum property of geodesics	171
7.5	Existence of geodesics	173
7.6	Deformation of horizontal arcs	174
CHAPTER V		
	Teichmüller Spaces	175
	Introduction to Chapter V	175
1.	Quasiconformal Mappings of Riemann Surfaces.	176
1.1	Complex dilatation on Riemann surfaces	176
1.2	Conformal structures	178
1.3	Group isomorphisms induced by quasiconformal mappings.	178
1.4	Homotopy modulo the boundary.	180
1.5	Quasiconformal mappings in homotopy classes	181
2.	Definitions of Teichmüller Space	182
2.1	Riemann space and Teichmüller space	182
2.2	Teichmüller metric.	183
2.3	Teichmüller space and Beltrami differentials	184
2.4	Teichmüller space and conformal structures.	185
2.5	Conformal structures on a compact surface	186
2.6	Isomorphisms of Teichmüller spaces	187
2.7	Modular group	188
3.	Teichmüller Space and Lifted Mappings.	189
3.1	Equivalent Beltrami differentials	189
3.2	Teichmüller space as a subset of the universal space	190
3.3	Completeness of Teichmüller spaces.	190
3.4	Quasi-Fuchsian groups	191
3.5	Quasiconformal reflections compatible with a group	192
3.6	Quasisymmetric functions compatible with a group	193
3.7	Unique extremality and Teichmüller metrics	195
4.	Teichmüller Space and Schwarzian Derivatives	196
4.1	Schwarzian derivatives and quadratic differentials	196
4.2	Spaces of quadratic differentials.	197
4.3	Schwarzian derivatives of univalent functions.	197
4.4	Connection between Teichmüller spaces and the universal space	198
4.5	Distance to the boundary.	200
4.6	Equivalence of metrics	201
4.7	Bers imbedding	203
4.8	Quasiconformal extensions compatible with a group	204
5.	Complex Structures on Teichmüller Spaces	205
5.1	Holomorphic functions in Banach spaces.	205

5.2	Banach manifolds	206
5.3	A holomorphic mapping between Banach spaces	207
5.4	An atlas on the Teichmüller space	208
5.5	Complex analytic structure	209
5.6	Complex structure under quasiconformal mappings	211
6.	Teichmüller Space of a Torus	212
6.1	Covering group of a torus	212
6.2	Generation of group isomorphisms	214
6.3	Conformal equivalence of tori	215
6.4	Extremal mappings of tori	216
6.5	Distance of group isomorphisms from the identity	218
6.6	Representation of the Teichmüller space of a torus	219
6.7	Complex structure of the Teichmüller space of torus	220
7.	Extremal Mappings of Riemann Surfaces	221
7.1	Dual Banach spaces	221
7.2	Space of integrable holomorphic quadratic differentials	222
7.3	Poincaré theta series	223
7.4	Infinitesimally trivial differentials	224
7.5	Mappings with infinitesimally trivial dilatations	226
7.6	Complex dilatations of extremal mappings	227
7.7	Teichmüller mappings	229
7.8	Extremal mappings of compact surfaces	231
8.	Uniqueness of Extremal Mappings of Compact Surfaces	232
8.1	Teichmüller mappings and quadratic differentials	232
8.2	Local representation of Teichmüller mappings	233
8.3	Stretching function and the Jacobian	235
8.4	Average stretching	236
8.5	Teichmüller's uniqueness theorem	237
9.	Teichmüller Spaces of Compact Surfaces	240
9.1	Teichmüller imbedding	240
9.2	Teichmüller space as a ball of the euclidean space	241
9.3	Straight lines in Teichmüller space	242
9.4	Composition of Teichmüller mappings	243
9.5	Teichmüller discs	244
9.6	Complex structure and Teichmüller metric	245
9.7	Surfaces of finite type	247
	Bibliography	248
	Index	253