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Graduate Texts in Mathematics

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John M. Lee

Riemannian Manifolds

An Introduction to Curvature

With 88 Illustrations

Springer

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To my family:
Pm, Nathan, and Jeremy Weizenbaum

Preface

This book is designed as a textbook for a one-quarter or one-semester graduate course on Riemannian geometry, for students who are familiar with topological and differentiable manifolds. It focuses on developing an intimate acquaintance with the geometric meaning of curvature. In so doing, it introduces and demonstrates the uses of all the main technical tools needed for a careful study of Riemannian manifolds.

I have selected a set of topics that can reasonably be covered in ten to fifteen weeks, instead of making any attempt to provide an encyclopedic treatment of the subject. The book begins with a careful treatment of the machinery of metrics, connections, and geodesics, without which one cannot claim to be doing Riemannian geometry. It then introduces the Riemann curvature tensor, and quickly moves on to submanifold theory in order to give the curvature tensor a concrete quantitative interpretation. From then on, all efforts are bent toward proving the four most fundamental theorems relating curvature and topology: the Gauss–Bonnet theorem (expressing the total curvature of a surface in terms of its topological type), the Cartan–Hadamard theorem (restricting the topology of manifolds of nonpositive curvature), Bonnet's theorem (giving analogous restrictions on manifolds of strictly positive curvature), and a special case of the Cartan–Ambrose–Hicks theorem (characterizing manifolds of constant curvature).

Many other results and techniques might reasonably claim a place in an introductory Riemannian geometry course, but could not be included due to time constraints. In particular, I do not treat the Rauch comparison theorem, the Morse index theorem, Toponogov's theorem, or their important applications such as the sphere theorem, except to mention some of them in passing; and I do not touch on the Laplace–Beltrami operator or Hodge theory, or indeed any of the multitude of deep and exciting applications of partial differential equations to Riemannian geometry. These important topics are for other, more advanced courses.

The libraries already contain a wealth of superb reference books on Riemannian geometry, which the interested reader can consult for a deeper treatment of the topics introduced here, or can use to explore the more esoteric aspects of the subject. Some of my favorites are the elegant introduction to comparison theory by Jeff Cheeger and David Ebin [CE75] (which has sadly been out of print for a number of years); Manfredo do Carmo's much more leisurely treatment of the same material and more [dC92]; Barrett O'Neill's beautifully integrated introduction to pseudo-Riemannian and Riemannian geometry [O'N83]; Isaac Chavel's masterful recent introductory text [Cha93], which starts with the foundations of the subject and quickly takes the reader deep into research territory; Michael Spivak's classic tome [Spi79], which can be used as a textbook if plenty of time is available, or can provide enjoyable bedtime reading; and, of course, the "Encyclopaedia Britannica" of differential geometry books, *Foundations of Differential Geometry* by Kobayashi and Nomizu [KN63]. At the other end of the spectrum, Frank Morgan's delightful little book [Mor93] touches on most of the important ideas in an intuitive and informal way with lots of pictures—I enthusiastically recommend it as a prelude to this book.

It is not my purpose to replace any of these. Instead, it is my hope that this book will fill a niche in the literature by presenting a selective introduction to the main ideas of the subject in an easily accessible way. The selection is small enough to fit into a single course, but broad enough, I hope, to provide any novice with a firm foundation from which to pursue research or develop applications in Riemannian geometry and other fields that use its tools.

This book is written under the assumption that the student already knows the fundamentals of the theory of topological and differential manifolds, as treated, for example, in [Mas67, chapters 1–5] and [Boo86, chapters 1–6]. In particular, the student should be conversant with the fundamental group, covering spaces, the classification of compact surfaces, topological and smooth manifolds, immersions and submersions, vector fields and flows, Lie brackets and Lie derivatives, the Frobenius theorem, tensors, differential forms, Stokes's theorem, and elementary properties of Lie groups. On the other hand, I do not assume any previous acquaintance with Riemannian metrics, or even with the classical theory of curves and surfaces in $\mathbf{R}^3$. (In this subject, anything proved before 1950 can be considered "classical.") Although at one time it might have been reasonable to expect most mathematics students to have studied surface theory as undergraduates, few current North American undergraduate math majors see any differential geometry. Thus the fundamentals of the geometry of surfaces, including a proof of the Gauss–Bonnet theorem, are worked out from scratch here.

The book begins with a nonrigorous overview of the subject in Chapter 1, designed to introduce some of the intuitions underlying the notion of curvature and to link them with elementary geometric ideas the student has seen before. This is followed in Chapter 2 by a brief review of some background material on tensors, manifolds, and vector bundles, included because these are the basic tools used throughout the book and because often they are not covered in quite enough detail in elementary courses on manifolds. Chapter 3 begins the course proper, with definitions of Riemannian metrics and some of their attendant flora and fauna. The end of the chapter describes the constant curvature "model spaces" of Riemannian geometry, with a great deal of detailed computation. These models form a sort of *leitmotif* throughout the text, and serve as illustrations and testbeds for the abstract theory as it is developed. Other important classes of examples are developed in the problems at the ends of the chapters, particularly invariant metrics on Lie groups and Riemannian submersions.

Chapter 4 introduces connections. In order to isolate the important properties of connections that are independent of the metric, as well as to lay the groundwork for their further study in such arenas as the Chern–Weil theory of characteristic classes and the Donaldson and Seiberg–Witten theories of gauge fields, connections are defined first on arbitrary vector bundles. This has the further advantage of making it easy to define the induced connections on tensor bundles. Chapter 5 investigates connections in the context of Riemannian manifolds, developing the Riemannian connection, its geodesics, the exponential map, and normal coordinates. Chapter 6 continues the study of geodesics, focusing on their distance-minimizing properties. First, some elementary ideas from the calculus of variations are introduced to prove that every distance-minimizing curve is a geodesic. Then the Gauss lemma is used to prove the (partial) converse—that every geodesic is locally minimizing. Because the Gauss lemma also gives an easy proof that minimizing curves are geodesics, the calculus-of-variations methods are not strictly necessary at this point; they are included to facilitate their use later in comparison theorems.

Chapter 7 unveils the first fully general definition of curvature. The curvature tensor is motivated initially by the question of whether all Riemannian metrics are locally equivalent, and by the failure of parallel translation to be path-independent as an obstruction to local equivalence. This leads naturally to a qualitative interpretation of curvature as the obstruction to flatness (local equivalence to Euclidean space). Chapter 8 departs somewhat from the traditional order of presentation, by investigating submanifold theory immediately after introducing the curvature tensor, so as to define sectional curvatures and give the curvature a more quantitative geometric interpretation.

The last three chapters are devoted to the most important elementary global theorems relating geometry to topology. Chapter 9 gives a simple moving-frames proof of the Gauss–Bonnet theorem, complete with a careful treatment of Hopf's rotation angle theorem (the *Umlaufsatz*). Chapter 10 is largely of a technical nature, covering Jacobi fields, conjugate points, the second variation formula, and the index form for later use in comparison theorems. Finally in Chapter 11 comes the *dénouement*—proofs of some of the "big" global theorems illustrating the ways in which curvature and topology affect each other: the Cartan–Hadamard theorem, Bonnet's theorem (and its generalization, Myers's theorem), and Cartan's characterization of manifolds of constant curvature.

The book contains many questions for the reader, which deserve special mention. They fall into two categories: "exercises," which are integrated into the text, and "problems," grouped at the end of each chapter. Both are essential to a full understanding of the material, but they are of somewhat different character and serve different purposes.

The exercises include some background material that the student should have seen already in an earlier course, some proofs that fill in the gaps from the text, some simple but illuminating examples, and some intermediate results that are used in the text or the problems. They are, in general, elementary, but they are *not optional*—indeed, they are integral to the continuity of the text. They are chosen and timed so as to give the reader opportunities to pause and think over the material that has just been introduced, to practice working with the definitions, and to develop skills that are used later in the book. I recommend strongly that students stop and do each exercise as it occurs in the text before going any further.

The problems that conclude the chapters are generally more difficult than the exercises, some of them considerably so, and should be considered a central part of the book by any student who is serious about learning the subject. They not only introduce new material not covered in the body of the text, but they also provide the student with indispensable practice in using the techniques explained in the text, both for doing computations and for proving theorems. If more than a semester is available, the instructor might want to present some of these problems in class.

Acknowledgments: I owe an unpayable debt to the authors of the many Riemannian geometry books I have used and cherished over the years, especially the ones mentioned above—I have done little more than rearrange their ideas into a form that seems handy for teaching. Beyond that, I would like to thank my Ph.D. advisor, Richard Melrose, who many years ago introduced me to differential geometry in his eccentric but thoroughly enlightening way; Judith Arms, who, as a fellow teacher of Riemannian geometry at the University of Washington, helped brainstorm about the "ideal contents" of this course; all my graduate students at the University of Washington who have suffered with amazing grace through the flawed early drafts of this book, especially Jed Mihalisin, who gave the manuscript a meticulous reading from a user's viewpoint and came up with numerous valuable suggestions; and Ina Lindemann of Springer-Verlag, who encouraged me to turn my lecture notes into a book and gave me free rein in deciding on its shape and contents. And of course my wife, Pm Weizenbaum, who contributed professional editing help as well as the loving support and encouragement I need to keep at this day after day.

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